A NOTE ON SOME k-FUNCTIONS AND THEIR PROBABILITY DISTRIBUTIONS

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ABSTRACT. *In this note, we define some new probability distributions involving a new parameter $k>0$, named as probability k-distributions. We prove some properties of these k-distributions which generalize the classical results. Here, we present the moment generating functions of said distributions and establishes the results in terms of Pochhammer k-symbol. Also, the authors prove some results showing the link between these distributions.*

1. INTRODUCTION

The history of k-functions theory is not very old. The theory started in the early years of 21st century when Diaz and Pariguan [1] introduced the generalized gamma k-function as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \ \ k>0, x\in\mathbb{C}\backslash k\mathbb{Z}^- \quad (1)$$

and also gave the properties of the said function. The Γ_k is one parameter deformation of the classical gamma function such that $\Gamma_k \to \Gamma$ as $k\to 1$. The Γ_k is based on the repeated appearance of the expression of the form

$$a(a+k)(a+2k)(a+3k)\ldots(a+(n-1)k). \quad (2)$$

The function of the variable a given by the statement (2), denoted by $(a)_{n,k}$ is called Pochhammer k-symbol. We obtain the usual Pochhammer symbol $(a)_n$ by taking $k=1$. The definition given in relation (1), is the generalization of $\Gamma(x)$ and the integral form of Γ_k is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, \qquad Re(x)>0. \quad (3)$$

From relation (3), we can easily show that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right). \quad (4)$$

The same authors defined the beta k-function as

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \ Re(x)>0, Re(y)>0 \quad (5)$$

and the integral form of $B_k(x,y)$ is

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\,dt. \quad (6)$$

From the definition of $B_k(x,y)$ given in relations (5) and (6), we can easily prove that

$$B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right). \quad (7)$$

Also, the researchers [2-7] have worked on the generalized gamma and beta k-functions and discussed their following properties:

$$\Gamma_k(x+k) = x\Gamma_k(x), \quad (8)$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \quad (9)$$

$$\Gamma_k(k) = 1, \ k>0, \quad (10)$$

$$B_k(x+k,y) = \frac{x}{x+y}B_k(x,y), \quad (11)$$

$$B_k(x,y+k) = \frac{y}{x+y}B_k(x,y), \quad (12)$$

$$B_k(xk,yk) = \frac{1}{k}B(x,y), \quad (13)$$

$$B_k(x,k) = \frac{1}{x}, \quad B_k(k,y) = \frac{1}{y}. \quad (14)$$

Note that when $k\to 1$, $B_k(x,y)\to B(x,y)$ and $\Gamma_k\to\Gamma$.

For more details about the theory of special k- functions like, gamma k- function, polygamma k-function, beta k- function, hypergeometric k- functions, solutions of k- hypergeometric differential equations, contegious functions relations, inequalities with applications and integral representations with applications involving gamma and beta k- functions, gamma and beta probability k- distributions and so forth (see [8-21]).

If a variable X assume all the values between two certain limits a and b, then x is said to be continuous variable, otherwise it is called a discrete variable. The distribution of the continuous variable is called continuous distribution and otherwise discrete distribution. If the frequencies of the variate (variable) are uniformly distributed about the mean position, then such distribution is known as symmetrical distribution and graph of such distribution is a symmetrical curve i.e., the values equidistant from the mean position have equal frequencies. If distribution is not symmetrical, it is said to be skewed and degree of its departure from the symmetry is termed as skewness. Actually, the skewness is the lack of symmetry.

Karl Pearson [22] introduced the term kurtosis to measure the degree of peakedness of frequency curve. When the values of the variable are closely bunched around the mode in such a way that the peak of the curve becomes relatively high, we say that the curve is Leptokurtic. If the curve has flat topped, it is a Platykurtic and if it is neither very high nor flat topped, then it is a normal curve or Mesokurtic. A moment designates the power to which the deviations are raised before averaging them. A link between the moments about arbitrary mean and actual mean of the data can be established in the following results.

$$\mu_r = \binom{r}{0}\mu_r' - \binom{r}{1}\mu_{r-1}'\mu_1' + \binom{r}{2}\mu_{r-2}'\mu_1'^2 - \binom{r}{3}\mu_{r-3}'\mu_1'^3 + \cdots. \quad (15)$$

Conversely, we have

$$\mu_r' = \binom{r}{0}\mu_r + \binom{r}{1}\mu_{r-1}\mu_1' + \binom{r}{2}\mu_{r-2}\mu_1'^2 + \binom{r}{3}\mu_{r-3}\mu_1'^3 + \cdots. \quad (16)$$

Remarks: From the above discussion, we observed that the first moment about the mean position is always zero while the second moment is equal to the variance.

In a random experiment with n outcomes, suppose a variable X assumes the values $x_1,\ldots,x_n$ with corresponding probabilities $p_1,\ldots,p_n$, then this collections is called probability distribution and $\sum p_i = 1$ (in case of discrete distributions). Also, if $f(x)$ is a continuous probability distribution function defined on an interval $[a,b]$, then

$\int_a^b f(x)dx = 1$. The expected value of the variate is defined as the first moment of the probability distribution about $x = 0$ i.e.,

$$\mu_1' = E(X) = \int_a^b xf(x)dx \quad (17)$$

and the rth moment about mean of the probability distribution is defined as $E(X-\mu)^r$, where μ is the mean of the distribution. Let $f(x)$ be the probability density function of a variate X in the distribution, then the expected value of e^{tX} is called moment generating function of the distribution about $x = 0$ and is denoted by $M_0(t)$, where t is a positive real number independent of x. Thus, for a continuous random variable X assuming all the values from a to b, the $m.g.f.$ is given by

$$M_0(t) = E(e^{tX}) = \int_a^b xf(x)dx. \quad (18)$$

2. Main Results: Gamma k-Distribution

Here, we introduce some probability distributions involving a new parameter $k > 0$. The use of this parameter generalizes the results available already in the literature. We give the definitions and some properties of these distributions and when $k = 1$, we get the classical results regarding these distributions.

(2.1) Definition: Let X be a continuous random variable, then it is said to have a gamma k-distribution with parameters $m > 0$ and $k > 0$, if its probability density k-function $(pdkf)$ is defined by [23]

$$f_k(x) = \begin{cases} \frac{1}{\Gamma_k(m)} x^{m-1} e^{\frac{-x^k}{k}}, & 0 \le x < \infty,\ k > 0 \\ 0, & \text{elsewhere} \end{cases} \quad (19)$$

and its distribution function$F_k(x)$ is defined by

$$F_k(x) = \begin{cases} \int_0^x \frac{1}{\Gamma_k(m)} z^{m-1} e^{\frac{-z^k}{k}} dz, & z > 0 \\ 0, & \text{elsewhere.} \end{cases} \quad (20)$$

Remarks: we can call the above function an incomplete gamma k- function because, if $k = 1$, it is an incomplete gamma function tabulated in [24-25].

(2.2) Proposition: The gamma k-distribution, satisfies the following properties for the parameters $m > 0$and $k > 0$.

(i): The gamma k- distribution is a proper probability distribution.

(ii): Mean of the gamma k- distribution is equal to a parameter m.

(iii): Variance of the gamma k-distribution in terms ok k is equal to mk.

(iv): The sum of two independent gamma k-distributions with parameters $m > 0$ and $n > 0$ is a gamma k-distribution with parameter $m + n$.

(v): The Harmonic mean of $\Gamma_k(m)$ variate in terms of k is $(m-k)$.

(vi): Mode of the gamma k- distribution exists at $x = (m-1)^{1/k}$ provided $m > 1, k > 0$.

Proof: For the proof of properties $(\mathrm{i}-\mathrm{iii})$(see[23]).

(iv): Let X and Y be two independent gamma k- variates with parameters $m > 0$ and $n > 0$ respectively, then

$$M_{x,k}(t) = (1-kt)^{-\frac{m}{k}},$$

$$and \quad M_{y,k}(t) = (1-kt)^{-\frac{n}{k}}.$$

Also, for two independent variates, the moment generating function of the sum is equal to the product of their moment generating function. Hence, we have

$$M_{(x+y),k}(t) = M_{x,k}(t)M_{y,k}(t)$$

$$= (1-kt)^{-\frac{m}{k}}(1-kt)^{-\frac{n}{k}}$$

$$= (1-kt)^{-\frac{(m+n)}{k}}.$$

Remarks: The result can be generalized up to a finite number of variates.

(v): Let X be a $\Gamma_k(m)$ variate, then we have the expected value of $\frac{1}{X}$ for $0 < x < \infty$ as

$$E\left(\frac{1}{X}\right) = \frac{1}{\Gamma_k(m)} \int_0^\infty \frac{1}{x} x^{m-1} e^{-\frac{x^k}{k}} dx$$

$$= \frac{1}{\Gamma_k(m)} \int_0^\infty x^{m-2} e^{-\frac{x^k}{k}} dx$$

$$= \frac{\Gamma_k(m-k)}{\Gamma_k(m)} = \frac{1}{m-k}$$

Now, Harmonic mean in terms of $k > 0$, is given by

$$H.M. = E_k\left(\frac{1}{1/X}\right) = m - k.$$

(vi): As defined earlier, the probability function of gamma k- distribution is

$$f_k(x) = \frac{1}{\Gamma_k(m)} x^{m-1} e^{-\frac{x^k}{k}}.$$

The mode occurs where frequency distribution is maximum. For this purpose, we proceed for maximum and minimum values as

$$f_k'(x) = e^{-\frac{x^k}{k}} \frac{1}{\Gamma_k(m)} x^{m-2}[m-1-x^k] = 0$$

$$\Rightarrow x = (m-1)^{1/k}.$$

and at $x = (m-1)^{1/k}$, we have

$$f_k''(x) = -k(m-1)(m-1)^{\frac{m-3}{k}}.$$

As $k > 0$, so $f_k''(x) < 0$ when $m > 1$. Thus, we have a maxima at $x = (m-1)^{1/k}$.

Remarks: The curve of gamma k- distribution is asymptotic to the x-axis. If $m > 1$, the curve has a mode at $x = (m-1)^{\frac{1}{k}}$, $k > 0$. If $m > 2k$, it touches the x-axis at the origin. The curve becomes asymptotic to both the axes when m lies between 0 and 1.

(2.3) Proposition: For $k > 0$, the moments generating function of gamma k- distribution is

$$\mu_{r,k}' = m(m+k)(m+2k)\ldots(m+(r-1)k)$$

$$= (m)_{r,k} \quad (21)$$

where $(m)_{r,k}$ is the Pochhammer k-symbol.

Proof: Using the definition of expected values along with the gamma k- distribution defined in the relation (19), the rth moments about $x = 0$ are given by

$$\mu_{r,k}' = E_k(X^r) = \int_0^\infty x^r \frac{1}{\Gamma_k(m)} x^{m-1} e^{-\frac{t^k}{k}} dx$$

$$= \frac{1}{\Gamma_k(m)} \int_0^\infty x^{m+r-1} e^{-\frac{t^k}{k}} dx$$

$$= \frac{\Gamma_k(m+rk)}{\Gamma_k(m)}$$
$$= \frac{m+(r-1)k\ldots\ (m+2k)(m+k)m\Gamma_k(m)}{\Gamma_{k(m)}}$$
$$= m(m+k)(m+2k)\ldots(m+(r-1)k).$$

To prove the second part of proposition (2.3), just use the relation $(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}$.

Remarks: When $r=1$, we obtain $\mu'_{1,k} = m$ = mean, when $r=2$, $\mu'_{2,k} = m(m+k)$ and hence $\mu_{2,k} = \mu'^2_{1,k} - \mu'_{2,k} = mk$= variance of the gamma k- distribution proved in the properties (2.2).

(2.4) Theorem: The gamma k- distribution is extremely skewed.

Proof: From the moment generating function of the gamma k- distribution, we see that

$\mu'_{1,k} = m,\qquad \mu'_{2,k} = m(m+k)$
$\mu'_{3,k} = m(m+k)(m+2k),$
$\mu'_{4,k} = m(m+k)(m+2k)(m+3k).$

Now, the moments about mean i.e., central moments in terms of k symbol are found as

$$\mu_{1,k} = 0$$
$$\mu_{2,k} = \mu'_{2,k} + (\mu'_{1,k})^2 = m(m+k) - m^2 = mk$$
$$\mu_{3,k} = \mu'_{3,k} - 3\mu'_{2,k}\mu'_{1,k} + 2(\mu'_{1,k})^3 = 2mk^2$$
$$\mu_{4,k} = \mu'_{4,k} - 4\mu'_{3,k}\mu'_{1,k} + 6\mu'_{2,k}(\mu'_{1,k})^2 - 3(\mu'_{1,k})^4$$
$$= 3mk^2(m+2k)$$

and

$$\beta_1 = \frac{\mu^2_{3,k}}{\mu^3_{2,k}} = \frac{4m^2k^4}{m^3k^3} = 4\frac{k}{m},$$
$$\beta_2 = \frac{\mu_{4,k}}{\mu^2_{2,k}} = \frac{3mk^2(m+2k)}{m^2k^2} = 3+6\frac{k}{m}.$$

As $k>0$ and $m>0$, so $\beta_1 \neq 0$ implies that k- distribution is not symmetrical. Also, $\beta_2 > 3$ implies that it is skewed and in particular, it is leptokurtic.

3. Beta k-Distribution

(3.1) Definition: Let X be a continuous random variable, then it is said to have a beta k- distribution of the first kind with two parameters m and n, if its probability density k- function $(pdkf)$ is defined by [23]

$$f_k(x) = \begin{cases} \frac{1}{kB_k(m,n)} x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}, 0 \le x \le 1;\ m,\ n,\ k>0 \\ 0, \qquad \text{elsewhere.} \end{cases} \quad (22)$$

In the above distribution, the k - beta variable of the first kind is referred to as $\beta_{1,k}(m,n)$ and its k - distribution function $F_k(x)$ is given by

$$F_k(x) = \begin{cases} 0, & x<0 \\ \int_0^1 \frac{1}{kB_k(m,n)} x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}dx, & 0\le x\le 1;\ m,n,k>0 \\ 0, & x>1 \end{cases} \quad (23)$$

Remarks: We can call the above function an incomplete k-beta function because, if $k=1$, it is an incomplete beta function tabulated in [24,26]

(3.2) Proposition: The beta k- distribution $B_{1,k}(m,n)m,n,k>0$ satisfies the following properties:

(i): Beta k- distribution is a proper probability k-distribution.

(ii): The mean of this distribution is $\frac{m}{m+n}$.

(iii): The variance of $B_{1,k}(m,n)$ in terms of k is $\frac{mnk}{(m+n)^2(m+n+k)}$.

(iv): The Harmonic mean of $B_{1,k}(m,n)$ in terms of k is $\frac{m-k}{m+n-k}$.

(v): Mode of the beta k- distribution of the first kind with parameters m and n exists at $x = \frac{m-k}{m+n-2k}$ provided that $m,n>k$.

Proof: Properties $(\mathrm{i}-\mathrm{iii})$ are proved in [23].

(iv): Let X be a $B_{1,k}(m,n)$ variate, then we have the expected value of $\frac{1}{X}$ as

$$E_k\left(\frac{1}{X}\right) = \frac{1}{kB_{1,k}(m,n)}\int_0^1 \frac{1}{x}x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}dx$$
$$= \frac{1}{kB_{1,k}(m,n)}\int_0^1 x^{\frac{m}{k}-1-1}(1-x)^{\frac{n}{k}-1}dx.$$

Using the relation (5) and (6), we get

$$E_k\left(\frac{1}{X}\right) = \frac{B_k(m-k,n)}{B_k(m,n)}$$
$$= \frac{\Gamma_k(m-k)\Gamma_k(n)}{\Gamma_k(m+n-k)}\frac{\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)}$$
$$= \frac{m+n-k}{m-k}.$$

Now, Harmonic mean in terms of k-symbol is given by

$$H.M. = E_k\left(\frac{1}{1/X}\right) = \frac{m-k}{m+n-k}.$$

(v): As defined earlier, the probability function of beta k- distribution of first kind is

$$f_k(x) = \frac{1}{kB_k(m,n)}x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}.$$

The mode occurs where frequency k- distribution is maximum. For this purpose, we find the maximum values as

$$f'_k(x) = \frac{1}{kB_k(m,n)}x^{\frac{m}{k}-2}(1-x)^{\frac{n}{k}-2}\frac{1}{k}[m-k-mx-nx+2kx].$$

Putting $f'_k(x)=0$ implies that $x=0$, $x=1$ and $x=\frac{m-k}{m+n-2k}$, which are the turning points of the function. Now,

$$f''_k(x) = -\frac{1}{kB_k(m,n)}x^{\frac{m}{k}-3}(1-x)^{\frac{n}{k}-3}\left[\frac{(m-k)(n-k)}{(m+n-2k)}\right].$$

At $x=0$ and $x=1$, $f''_k(x)=0$. As $k>0$, so $f''_k(x)<0$ when $m,n>k$. Thus, we have maxima at $x=\frac{m-k}{m+n-2k}$ which is the model value.

Remarks: If $m,n>k>0$, the k- distribution has a model value at $x=\frac{m-k}{m+n-2k}$. If $m=n=k$, it reduces to the uniform distribution over the unit interval. The curve touches the x-axis at $x=0$ when $m>2k$.

(3.3) Theorem: The rth moment of the beta k- distribution of first kind is given by $\frac{(m)_{r,k}}{(m+n)_{r,k}}$.

Proof: The rth moment in terms of $k>0$, about the origin i.e., about $x=0$ are

$$\mu'_{r,k} = E_k(X^r)$$
$$= \frac{1}{kB_k(m,n)}\int_0^1 x^r.x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}dx$$

$$= \frac{1}{kB_k(m,n)} \int_0^1 x^{\frac{m}{k}+r-1}(1-x)^{\frac{n}{k}-1} dx$$

$$= \frac{B_k(m+rk,\ n)}{B_k(m,\ n)}$$

$$= \frac{\Gamma_k(m+rk)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(m+rk+n)}.$$

Using the relation $(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}$ in numerators as well as in denominator, we get desired result.

(3.4) Definition: A continuous random variable X is said to have a beta k- distribution of the second kind with parameters $m$and n, if its probability density k- function $(pdkf)$ is define by

$$f_k(x) = \begin{cases} \frac{1}{kB_k(m,n)} \frac{x^{\frac{m}{k}-1}}{(1+x)^{\frac{m+n}{k}}} & 0 \le x < \infty;\ m,n,k > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (24)$$

In the above k- distribution, the k- beta variable of the second kind is referred to as $B_{2,k}(m,n)$ and its k- distribution function $F_k(x)$ is given by

$$F_k(x) = \begin{cases} \int_0^\infty \frac{1}{kB_k(m,n)} \frac{x^{\frac{m}{k}-1}}{(1+x)^{\frac{m+n}{k}}} dx & 0 \le x < \infty;\quad m,n,k > 0 \\ 0, & x < 0 \end{cases} \quad (25)$$

(3.5) Proposition: The beta k- distribution of the second kind $B_{2,k}(m,n)$ satisfies the following properties:

(i): The k- distribution is a proper probability distribution.

(ii): The mean of k- distribution is $\frac{m}{n-k}$.

(iii): The variance of $B_{2,k}(m,n)$ is $\frac{mk(m+n-k)}{(n-k)^2(n-2k).}$

(iv): Harmonic mean of $B_{2,k}(m,n)$ is terms of k is $\frac{m-k}{n}$.

(v): Mode of the beta k- distribution with parameters m and n exists at $x = \frac{m-k}{n+k}$ provided that $m > k$.

Proof: For the properties $(\mathrm{i}-\mathrm{iii})$(see [23]).

(iv): Let X be a $B_{2,k}(m,n)$variates, then we have the expected value of $\frac{1}{X}$ as

$$E_k\left(\frac{1}{X}\right) = \frac{1}{kB_k(m,n)} \int_0^\infty \frac{1}{x} \frac{x^{\frac{m}{k}-1}}{(1+x)^{\frac{m+n}{k}}} dx$$

$$= \frac{1}{kB_k(m,n)} \int_0^\infty x^{\frac{m}{k}-1-1}(1+x)^{\frac{m+n}{k}} dx.$$

Using the relation (5) and (6), we get

$$E_k\left(\frac{1}{X}\right) = \frac{B_k(m-k,n+k)}{B_k(m,n)}$$

$$= \frac{\Gamma_k(m-k)\Gamma_k(n+k)}{\Gamma_k(m+n)} \frac{\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)}$$

$$= \frac{n}{m-k}$$

Now, Harmonic mean in terms of $k > 0$, is given by

$$H.M. = E_k\left(\frac{1}{1/X}\right) = \frac{m-k}{n}.$$

(v): As defined earlier, the probability function of beta k- distribution of second kind is

$$f_k(x) = \frac{1}{kB_k(m,n)} \frac{x^{\frac{m}{k}-1}}{(1+x)^{\frac{m+n}{k}}}, \quad 0 \le x < \infty; \quad m,n,k > 0.$$

The mode occurs where frequency k-distribution is maximum. For this purpose, we find the maximum values as

$$f_k'(x) = \frac{1}{k^2B_k(m,n)} x^{\frac{m}{k}-2}(1+x)^{\frac{m+n}{k}-1}[m-k-kx-nx].$$

Putting $f_k'(x) = 0$ implies that $x = 0$ and $x = \frac{m-k}{n+k}$, which are the turning points of the function. At $x = 0$, $f_k''(x) = 0$ and at $x = \frac{m-k}{n+k}$,

$$f_k''(x) = -(k+n)\left(\frac{m+n}{n+k}\right)^{\frac{m+n-k}{k}} \left(\frac{m-k}{n+k}\right)^{\frac{m-2k}{k}} < 0$$

Which implies that modal value exists at $x = \frac{m-k}{n+k}$ provided that $m > k$.

Remarks: If $m > k > 0$, the k- distribution is unimodal with a mode at $x = \frac{m-k}{n+k}$. If $m = k$, the k- distribution is J-shaped. The curve is asymptotic to the x-axis and touches it at the origin when $m > 2k$. The curve touches the y- axis at the origin if $k > m > 2k$ and becomes asymptotic to both the axes when m lies between 0 and 1.

(3.6) Theorem: The moments of the higher order of beta k-distribution of the second kind are given as:

$$\mu'_{r,k} = \frac{m(m+k)(m+2k)\ldots(m+(r-1)k)}{(n-k)(n-2k)\ldots(n-rk)}$$

$$= (m)_{r,k}(n)_{-r,k}\,\Gamma_k(n).$$

Proof: Consider

$$\mu'_{r,k} = E_k(X^r) = \int_0^\infty \frac{1}{kB_k(m,n)} \frac{x^{\frac{m}{k}-1+r}}{(1+x)^{\frac{m+n}{k}}} dx.$$

Changing the variables as $x = \frac{1-y}{y} \Rightarrow dx = \frac{-1}{y^2}dy$ and above equation becomes

$$\mu'_{r,k} = \frac{1}{kB_k(m,n)} \int_0^1 y^{\frac{n}{k}-r-1}(1-y)^{\frac{m}{k}+r-1} dy.$$

Replacing $(1-y)$ by t, we have

$$\mu'_{r,k} = \frac{1}{B_k(m,n)} \frac{1}{k} \int_0^1 t^{\frac{m}{k}+r-1}(1-t)^{\frac{n}{k}-r-1} dt.$$

$$= \frac{B_k(m+rk,\ n-rk)}{B_k(m,n)}$$

$$= \frac{\Gamma_k(m+rk)\Gamma_k(n-rk)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)\Gamma_k(m+n)}$$

$$= \frac{\Gamma_k(m+rk)\Gamma_k(n-rk)}{\Gamma_k(m)\Gamma_k(n)}.$$

Now using $\Gamma_k(n-rk) = \frac{\Gamma_k(n)}{(n-k)(n-2k)\ldots(n-rk)}$ in the above equation, we get the first part of the desired result. If we use the relations $\Gamma_k(x+k) = x\Gamma_k(x)$ and $(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}$, then second part can be obtained easily.

(3.7)Theorem: If x and y are two independent gamma k-variates with parameters m and n respectively. Then the quotient $\frac{x}{x+y}$ is a k times $B_{1,k}$ variate of the first kind with parameters mk and nk.

Proof: Since x is a gamma k- variate with parameter m, therefore the probability differential of x is given by

$$dP(x) = \frac{e^{-\frac{x^k}{k}} x^{m-1}}{\Gamma_k(m)} dx. \quad (26)$$

Let $z = \frac{x}{x+y} \Rightarrow z = \frac{1}{1+y/x}$. Since x ranges from 0 to ∞, z ranges from 0 to1. Also, $x = \frac{zy}{1-z}$ and treating y as constant$\Rightarrow$ $dx = \frac{y}{(1-z)^2}\, dz$. Thus, equation (26) gives

$$dP(z) = \frac{e^{-\frac{(\frac{zy}{1-z})^k}{k}}}{\Gamma_k(m)} \left(\frac{yz}{1-z}\right)^{m-1} \frac{y}{(1-z)^2} dz.$$

$$= \frac{e^{-\frac{(\frac{zy}{1-z})^k}{k}}}{\Gamma_k(m)} \frac{y^m z^{m-1}}{(1-z)^{m+1}} dz.$$

Again, y is a gamma k- variate with parameter n. So, the probability differential of y is

$$dP(y) = \frac{e^{-\frac{y^k}{k}} y^{n-1}}{\Gamma_k(n)} dy, \qquad 0 \le y < \infty$$

The probability differential $dP(y,z)$ of z and y to be in $[z-dz, z+dz]$ and $[y-dy, y+dy]$ simultaneously, is give by

$$dP(y,z) = dP(y).dP(z)$$

$$= \int_0^\infty \int_0^1 \frac{e^{-\frac{(\frac{zy}{1-z})^k}{k}}}{\Gamma_k(m)} \frac{y^m z^{m-1}}{(1-z)^{m+1}} \frac{e^{-\frac{y^k}{k}} y^{n-1}}{\Gamma_k(n)}\, dzdy$$

$$= \frac{1}{\Gamma_k(m)\Gamma_k(n)} \int_0^\infty \int_0^1 e^{-\frac{(\frac{zy}{1-z}+y)^k}{k}} y^{m+n-1} \frac{z^{m-1}}{(1-z)^{m+1}} dzdy.$$

By changing the variable as $z = s$ and $y = t(1-s)$, we have $\frac{zy}{1-z} = t$ and $0 \le y < \infty$, $0 \le z \le 1$ gives $0 \le t < \infty$ and $0 \le s \le 1$. The Jacobian of this transformation is (see [27])

$$\frac{\partial(y,z)}{\partial(s,t)} = \begin{vmatrix} 0 & 1 \\ 1-s & -t \end{vmatrix} = -(1-s).$$

Now, we conclude that $dzdy = \left|\frac{\partial(y,z)}{\partial(s,t)}\right| dsdt$. Thus, we have

$$dP(y,z) =$$

$$\frac{1}{\Gamma_k(m)\Gamma_k(n)} \int_0^\infty e^{-\frac{t^k}{k}} t^{m+n-1} dt \int_0^1 s^{m-1}(1-s)^{n-1} ds$$

$$= \frac{\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)} \int_0^1 s^{\frac{km}{k}-1}(1-s)^{\frac{kn}{k}-1} ds$$

$$= k\, \frac{1}{\mathrm{B}_k(m,n)} \frac{1}{k} \int_0^1 s^{\frac{km}{k}-1}(1-s)^{\frac{kn}{k}-1} ds.$$

(3.8) Theorem: If x is beta k- variate of the second kind with parameters m and n, then its reciprocal is also beta k- variate of the same kind with parameters unchanged.

Proof: Let x be a $B_{2,k}$ variate, then its probability differential is

$$dP(x) = \frac{x^{\frac{m}{k}-1}}{B_{2,k}(m,n)(1+x)^{m+n}} dx, \quad 0 \le x < \infty. \qquad (27)$$

Put $x = \frac{1}{y} \Rightarrow dx = \frac{-1}{y^2} dy$ and equation (27), give

$$dP(y) =$$

$$\frac{\left(\frac{1}{y}\right)^{\frac{m}{k}-1}}{B_{2,k}(m,n)\left(1+\frac{1}{y}\right)^{m+n}} \left(-\frac{1}{y^2}\right) dy, \quad 0 \le y < \infty.$$

$$= \frac{(y)^{\frac{m}{k}-1}}{B_{2,k}(m,n)(1+y)^{m+n}}\, dy \quad, \quad 0 \le y < \infty.$$

Which is a $B_{2,k}(m,n)$ variate with parameters unchanged.

(3.9) Theorem: For each beta k- variate of the first kind, there exists a pair of beta k- variate of the second kind and conversely.

Proof: Let v be a $B_{1,k}$ variate with parameters m and n, then its probability differential is

$$dP(v) = \frac{v^{\frac{m}{k}-1}(1-v)^{\frac{n}{k}-1}}{B_{1,k}(m,n)} dv, \qquad 0 \le v \le 1.$$

Changing the variable as $v = \frac{1}{w}$, we have

$$dP(w) = \frac{w^{\frac{m}{k}-1}}{B_{2,k}(m,n)(1+w)^{m+n}} dw, \qquad 0 \le w < \infty.$$

Which implies that w is $B_{2,k}(m,n)$ variate. Also, reciprocal of w will be another variate (Proved earlier).

Conversely, let w be the $B_{2,k}(m,n)$ variate, then probability differential is

$$dP(w) = \frac{w^{\frac{m}{k}-1}}{B_{2,k}(m,n)(1+w)^{m+n}} dw, \qquad 0 \le w < \infty.$$

We can prove the result by putting $v = \frac{1}{w}$ and $w = 1 - v$ in the above equation.

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