ON DISTRIBUTIVE AG-GROUPOIDS

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ABSTRACT: A groupoid is called an AG-groupoid if it satisfies the left invertive law: $a \cdot bc = c \cdot ba$. We discuss the notion of distributive AG-groupoids. We give their enumeration up to order 6 and then analyze our data to derive some interesting relations of these AG-groupoids to other subclasses of AG- groupoids. We also present a few conjectures for distributive groupoids in general.

Keywords: left distributive, right distributive, AG-groupoid, counting.

1. INTRODUCTIONAND PRELIMINARIES

The structure of AG-groupoids is one of the interesting non-associative structures. In fact, AG-groupoids generalize commutative semigroups. Kepka et al [13] studied distributive groupoids. We study distributive AG-groupoids in this paper. AG-groupoids were enumerated up to order 6 in [6]. Using GAP [7] we give from that data, the enumeration of distributive groupoids up to order 6 and then we analyze our data to find some interesting relations of these AG-groupoids to other subclasses of AG-groupoids. We were aided by Prover9 [5] in finding conjectures about these AG-groupoids which we then proved manually. All the examples and counter examples were constructed by Mace4 [5]. We also give a few conjectures for distributive groupoids in general as a future work.

We first recall that a groupoid is defined as a non-empty set S together with a binary operation $*: S \times S \to S$ [6]. In the sequel we will generally elide the binary operation. An element a of a groupoid S is called left cancellative if $ax = ay \Rightarrow x = y$ for all $x, y \in S$ [12, 15]. Similarly, an element a of a groupoid S is called right cancellative if $xa = ya \Rightarrow x = y$ for all $x, y \in S$ [4, 15]. An element a of agroupoid S is called caconsecutivef it is both left and right cancellative [14]. A groupoid S is called (left, right) cancellative if all of its elements are (left, right) cancellative [4,12]. A groupoid S is called idempotent groupoid if $aa = a \, \forall a \in S$ [12, 15]. A groupoid S is called medial (paramedial) if S satisfies the medial law (paramedical law),$ab \cdot cd = ac \cdot bd$ ($ab \cdot cd = db \cdot ca$) [9]. Recall that in [9] the name right modular groupoid has been used instead of AG-groupoid for a groupoid which satisfies the left invertive law: $a \cdot bc = c \cdot ba$. It has also been proved that an AG-groupoid is always medial and with left identity it becomes paramedical [14].

We now define various subclasses of AG-groupoid as follows:

An AG-groupoid S is called

—an AG-monoid if it has a left identity [6],

—an AG$*$-groupoid if it satisfies the identity: $ab \cdot c = b \cdot ac$ [12],

—an AG$**$-groupoid if it satisfies the identity: $a \cdot bc = b \cdot ac$ [6],

—a locally associative if it satisfies the identity: $a \cdot aa = aa \cdot a$ [4, 12],

—an AG-band if it satisfies the identity: $aa = a$ [4, 12, 15],

—an AG-3-band if it satisfies the identity: $a\,aa = aa\,a = a$ [4, 12],

—a paramedical if it is a paramedial groupoid [4. 15],

—a Bol$*$-groupoid if it satisfies the identity: $(ab \cdot c)d = a(bc \cdot d)$ [4, 12],

—a cancellative if it is either left cancellative or right cancellative [6],

—a flexible if it satisfies the identity: $a \cdot ba = ab \cdot a$ [4, 12],

—a left alternative if it satisfies the identity: $a \cdot ab = aa \cdot b$ [4, 12, 15],

—a right alternative if it satisfies the identity: $a \cdot bb = ab \cdot b$ [4, 12, 15],

—an alternative if it is both left alternative and right alternative [4, 12, 15],

—a unipotent if it satisfies the identity: $aa = bb$ [4, 12, 15],

—a T^1-AG-groupoid if $\forall a, b, c, d \in S, ab = cd \Rightarrow ba = dc$ [4, 12, 15],

—a T^2-AG-groupoid if $\forall a, b, c, d \in S, ab = cd \Rightarrow ac = bd$,

—a T_l^3-AG-groupoid if $\forall a, b, c, d \in S, ab = ac \Rightarrow ba = ca$ [6],

—a T_r^3-AG-groupoid if $\forall a, b, c, d \in S, ba = ca \Rightarrow ab = ac$ [6],

—a T^3-AG-groupoid if it is both T_l^3-AG-groupoidand T_r^3-AG-groupoid,

—a T_f^4-AG-groupoid if $\forall a, b, c, d \in S, ab = cd \Rightarrow ad = cb$,

—a T_b^4-AG-groupoid if $\forall a, b, c, d \in S, ab = cd \Rightarrow da = bc$,

—a T^4-AG-groupoid if it is both T_f^4-AG-groupoid and T_b^4-AG-groupoid.

The above definitions can be found in [2, 3, 8].

Definition 1, [4, 14]

(i) A groupoid S is called a left distributive groupoid if
$a \cdot bc = ab \cdot ac \; \forall a, b, c \in S$,

(ii) A groupoid S is called a right distributive groupoid if
$ab \cdot c = ac \cdot bc \; \forall a, b, c \in S$,

(iii) A groupoid S is called a distributive groupoid if it is both left distributive and right distributive.

2. ENUMERATION AND ANALYSIS OF DISTRIBUTIVE AG-GROUPOIDS

We start by the following definition.

Definition 2 [4, 14, 12, 15]

(i) An AG-groupoid S is called a left distributive AG-groupoid if $a \cdot bc = ab \cdot ac \forall a, b, c \in S$,

(ii) An AG-groupoid S is called a right distributive AG-groupoid if $ab \cdot c = ac \cdot bc \ \forall a, b, c \in S$,

(iii) An AG-groupoid S is called a distributive AG-groupoid if it is both left distributive and right distributive.

We will denote left distributive groupoid, right distributive groupoid and distributive groupoid by LD-groupoid, RD-groupoid and D-groupoid respectively. Similarly, we will denote left distributive AG-groupoid, right distributive AG-groupoid and distributive AG-groupoid by LD-AG-groupoid, RD-AG-groupoid and D-AG-groupoid respectively. Moreover, it will be better to denote left distributive semigroup, right distributive semigroup and distributive semigroup by LD-semigroup, RD-semigroup and D-semi group respectively.

It is enough to prove the existence of the last six of the above groupoids by examples because these imply the existence of the first three.

Example 1. The examples are given by:

(i) An LD-AG-groupoid,

(ii) An RD-AG-groupoid,

(iii) AD-AG-groupoid,

(iv) An LD-AG-semigroup

(v) An RD-AG-semigroup,

(vi) AD-AG-semigroup.

Kepka et al [13] proved the following important result about D-groupoids.

(i)

·	0	1	2	3
0	1	2	2	2
1	3	2	2	2
2	2	2	2	2
3	2	2	2	2

(ii)

·	0	1	2
0	1	1	1
1	2	2	2
2	2	2	2

(iii)

·	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

(iv)

·	0	1	2
0	0	1	2
1	2	2	2
2	2	2	2

(v)

·	0	1	2
0	2	0	2
1	2	1	2
2	2	2	2

(vi)

·	0	1
0	0	0
1	1	1

Table 1 presents the counting of the (left, right) distributive AG-groupoids. Note that only the number of non-associative AG-groupoids is shown.

Order	3	4	5	6
Total	20	331	31913	40104513
Left distributive AG-groupoids	0	2	31	
Right distributive AG-groupoids	6	176	21190	
Distributive AG-groupoids	0	1	4	

TABLE 1. Classification and enumeration results for (left, right) distributive AG-groupoids of orders 3–6.

Lemma 1. [13, Corollary1.8]. Every D-groupoid satisfies the following identities:

(i) $a \cdot ba = ab \cdot a$,

(ii) $aa \cdot bc = ab \cdot ac = a \cdot bc$

(iii) $bc \cdot a = ba \cdot ca = bc \cdot aa$.

Theorem 1 [4, 15]. Let S be a D-AG-groupoid. Then S is a D-semigroup if any of the following holds.

(i) S is left nuclear square,

(ii) S is middle nuclear square,

(iii) S is right nuclear square.

Proof. For all $a, b, c \in S$ by left invertive law we have, $ab \cdot c = cb \cdot a$. Which by using Lemma 1(ii), (iii) becomes, $ab \cdot c = (cb)^2 \cdot a^2$. Which by using medial law becomes, $ab \cdot c = c^2 b^2 \cdot a^2$. Again by using left invertive law [4] it becomes $ab \cdot c = a^2 b^2 \cdot c^2$. Which by definition of left nuclear square property or middle nuclear square property or right nuclear square property becomes, $ab \cdot c = a^2 \cdot b^2 c^2$. Again by using medial law it becomes $ab \cdot c = a^2 \cdot (bc)^2$. Finally, using Lemma 1(ii), (iii) it becomes, $ab \cdot c = a \cdot bc$. Hence S is a D-semigroup.

Corollary 1. Every nuclear square D-AG-groupoid is a semigroup.

Note that though a nuclear square D-AG-groupoid is associative but it is not necessarily commutative.

Example 2. A non-commutative nuclear square D-AG-groupoid of order 4:

·	0	1	2	3
0	2	2	2	2
1	3	2	2	2
2	2	2	2	2
3	2	2	2	2

Prover9 shows that Theorem1 and Corollary 1 hold in general for (left, middle, right) nuclear square D-AG-groupoid. However, we are unable to prove it. So we give it as a conjecture for future work.

Conjecture 1 [4, 12, 15]**.** Let S be a D-groupoid. Then S is a D-semigroup if any of the following holds.

(i) S is left nuclear square,

(ii) S is middle nuclear square,

(iii) S is right nuclear square,

(iv) S is nuclear square.

Note that both left distributivity and right distributivity are necessary for holding the last two parts of Lemma1.

Example 3. The counter examples are given by:

(i) An LD-groupoid which does not satisfy $aa \cdot bc = a \cdot bc$,

(ii) An RD-groupoid which does not satisfy $bc \cdot a = bc \cdot aa$

(iii) An LD-groupoid which does not satisfy $a \cdot ba = ab \cdot a$,and

(iv) An RD-groupoid which does not satisfy $a \cdot ba = ab \cdot a$.

(i)

·	0	1
0	1	1
1	0	1

(ii)

·	0	1
0	1	0
1	1	1

(iii)

·	0	1
0	1	0
1	1	0

(iv)

·	0	1
0	1	1
1	0	0

But in case of AG-groupoids, left distributivity and right distributivity are not necessary for holding the last two parts of Lemma1.

Lemma 2. Let S be a groupoid. Then

(i) $aa \cdot bc = a \cdot bc$ if S is LD-AG-groupoid.

(ii) $bc \cdot a = bc \cdot aa$ if S is RD-AG-groupoid.

Proof. Both parts follow by definition and medial law.

Lemma1 Part (i) does not hold for only LD-AG-groupoid or only RD-AG- groupoid.

Example 4. The counter examples are given by: (i) a non-flexible LD-AG-groupoid, and (ii) a non-flexible RD-AG groupoid.

(i)

·	0	1	2	3
0	1	2	2	2
1	3	2	2	2
2	2	2	2	2
3	2	2	2	2

(ii)

·	0	1	2
0	1	1	1
1	2	2	2
2	2	2	2

The following theorem collects some important implications.

Theorem 2. These implications always hold.

(i) AG∗-groupoid ⇒ Bol∗-AG-groupoid [1,Theorem1],

(ii) AG-monoid ⇒ AG∗∗-groupoid ⇒ Bol∗-AG-groupoid [8],

(iii) T^2-AG-groupoid ⇒T1-AG-groupoid [2],

(iv) T^4-AG-groupoid ⇒T1-AG-groupoid [2],

(v) T^1-AG-groupoid ⇒Bol∗-AG-groupoid [2],

(vi) Bol∗-AG-groupoid ⇒ paramedial AG-groupoid ⇒left nuclear square AG-groupoid [8],

(vii) Right alternative AG-groupoid ⇒ left nuclear square AG-groupoid [4, 12].

(viii) Unipotent AG-groupoid ⇒ left nuclear square AG-groupoid [4, 12, 15].

Proof. We just need to prove part (viii). Let S be a unipotent AG-groupoid. Let $a, b, c \in S$ [4, 12, 15]. Then by definition of unipotent AG-groupoid, we have $aa \cdot bc = cc \cdot bc$. This by medial law implies that $aa \cdot bc = cb \cdot cc$. By left invertive law the last equation becomes $aa \cdot bc = (cc \cdot b)c$. Finally once again by definition of unipotent AG-groupoid we get $aa \cdot bc = (aa \cdot b)c$. Hence $S$is left nuclear square.

Now from Theorem 1 and Theorem 2 we get the following corollary.

Corollary 2. AD-groupoid is D-semigroup if any of the following holds.

(i) S is an AG∗-groupoid [6],

(ii) S is an AG-monoid [6],

(iii) S is an AG∗∗-groupoid [6],

(iv) S is a T^4-AG-groupoid,

(v) S is a T^2-AG-groupoid,

(vi) S is a T^1-AG-groupoid,

(vii) S is a Bol∗-AG-groupoid,

(viii) S is a paramedial AG-groupoid,

(ix) S is a right alternative AG-groupoid,

(x) S is a unipotent AG-groupoid.

Again Prover9 shows that the following holds in general but we are unable to prove it. So we give it as our second conjecture.

Conjecture 2. Every Bol∗-D-groupoid is a D-semigroup.

By Corollary 2 (viii) a paramedial D-AG-groupoid is a D-semigroup. However a paramedial D-groupoid is not necessarily a D-semigroup. Also for a paramedial groupoid, LD-groupoid is not necessarily equivalent to RD-groupoid.

Example 5. The counter examples are given by:

(i) A paramedical D-groupoid which is not a D-semigroup,

(ii) A paramedial LD-groupoid which is not RD-groupoid [6],

(iii) A paramedial RD-groupoid which is not LD-groupoid [6].

(i)

·	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

(ii)

·	0	1	2
0	1	1	1
1	2	1	1
2	1	1	1

(iii)

·	0	1	2
0	1	1	2
1	2	2	2
2	2	2	2

By Corollary 2 (vi) a T^1-D-AG-groupoid is a D-semigroup. However, a T^1-D- groupoid is not necessarily a D-semigroup.

Example 6. A non-associative T1- D-groupoid of order 4:

·	0	1	2	3
0	0	2	1	0
1	2	1	0	2
2	1	0	2	1
3	0	2	1	0

However Prover9 suggests that the following holds in general.

Conjecture 3. A T^1-groupoid S is LD-groupoid iff it is RD-groupoid.

Consider the following theorem.

Theorem 3. A T^4-RD-AG-groupoid S [4] is a semigroup.

Proof. Let $a, b, c \in S$. By right distributivity $ab \cdot c = ac \cdot bc$[4]. By left invertive law [12] this gives $cb \cdot a = ac \cdot bc$. Since S is a T_b^4−AG-groupoid, so the last equation gives $bc \cdot cb = a \cdot ac$. Again using left invertive law this gives, $(cb \cdot c)b = a \cdot ac$. Since S is aT_f^4−AG-groupoid, we have $(cb \cdot c)(ac) = ab$. By right distributivity this gives, $(cb \cdot a)c = ab$. Again by definition of T_f^4−AG-groupoid [15], $(cb \cdot a)b = ac$. By left invertive law, we have $ba \cdot cb = ac$. Finally, by definition of T_b^4−AG-groupoid, $c \cdot ba = cb \cdot a$. Hence S is a semigroup.

Theorem 3 shows that we only consider the right distributivity for T^4-AG- groupoid to become semigroup. It looks from the proof of this theorem that there is no need of left distributivity. In fact, it is not so and left distributivity comes into it automatically as we will prove below. The actual result is as following.

Theorem 4. A T^4-D-AG-groupoid S is a D-semigroup.

Theorem 5. A semigroup S is LD-AG-groupoid iff it is RD-AG-groupoid [4, 12, 15].

Proof. Let S be an LD-AG-groupoid and let $a, b, c \in S$. By left invertive law, associativity, Lemma 2 (i) and medial law, we have $ab \cdot c = cb \cdot a = c \cdot ba = c^2 \cdot ba = c^2 b \cdot a = ab \cdot c^2 = ac \cdot bc$. Hence S is an RD-AG-groupoid [4, 12, 15].

Conversely, let S be an RD-AG-groupoid [4, 12, 15] and let $a, b, c \in S$. By associativity, left invertive law, Lemma2 (ii) and medial law, we have $a \cdot bc = ab \cdot c = cb \cdot a = cb \cdot a^2 = a^2 b \cdot c = a^2 \cdot bc = ab \cdot ac$. Hence S is an LD-AG-groupoid.

Remark 1. An associative AG-groupoid is not necessarily commutative for example see Example 2.

A non-associative flexible AG-groupoid can be only LD-AG-groupoid, only RD- AG-groupoid or can be D-AG-groupoid.

Example 7. The counter examples are given by:

(i) Non-associative flexible LD-AG-groupoid which is not RD-AG-groupoid [6],

(ii) non-associative flexible RD-AG-groupoid which is not LD-AG-groupoid [6]

(i)

·	0	1	2	3	4	5
0	2	2	4	4	4	4
1	3	3	4	4	4	4
2	4	5	4	4	4	4
3	5	4	4	4	4	4
4	4	4	4	4	4	4
5	4	4	4	4	4	4

(ii)

·	0	1	2
0	2	1	2
1	2	2	2
2	2	2	2

(iii)

·	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

(iii) non-associative flexible D-AG-groupoid.

Example 9 (iii) is also an example of a non-associative D-AG-band. We next prove that:

Theorem 6. An AG-band S is always D-AG-groupoid [4, 12].

Proof. Let $a, b, c \in S$. Then by using definition of AG-band and medial law, we have $ab \cdot c = ab \cdot c^2 = ac \cdot bc$. Hence S is an RD-AG-groupoid. Similarly we can show that S is an LD-AG-groupoid.

From Lemma1it easily follows that:

Theorem 7. A cancellative D-groupoid is idempotent groupoid.

For the detailed study of cancellative AG-groupoids we refer to [12]. The following example shows that Theorem 6 cannot be generalized to idempotent groupoids.

Example 8. The counter examples are given by:

(i) non-associative idempotent LD-groupoid which is not RD-groupoid, and

(ii) non-associative idempotent RD-groupoid which is not LD-groupoid.

(i)

·	0	1	2
0	0	2	1
1	1	1	1
2	2	2	2

(ii)

·	0	1	2
0	0	2	2
1	0	1	2
2	0	0	2

AG-3-band is the generalization of AG-band. AG-bands and AG-3-bands were studied in [10, 11].

Example 9 (iii) is also an example of a non-associative D-AG-3-band. But in contrast to AG-band, AG-3-band can be such that it has neither left distributivity nor right distributivity.

Example 9. A non-associative AG-3-band which is neither LD-AG-groupoid nor RD-AG-groupoid:

·	0	1	2	3	4	5
0	2	3	0	4	1	5
1	4	1	4	5	3	4
2	0	3	2	4	1	5
3	1	4	1	3	5	1
4	3	5	3	1	4	3
5	5	3	5	4	1	5

However we have the following result for AG-3-bands.

Theorem 8. An AG-3-band$S$ is LD-AG-groupoid iff it is RD-AG-groupoid.

Proof. Let S bean LD-AG-groupoid [4, 12, 15] and let $a, b, c \in S$. By definition of AG-3-band, left invertive law, Lemma 2 (i) and medial law, we have

$ab \cdot c = (ab)^2(ab) \cdot c = c(ab) \cdot (ab)^2 = c^2(ab) \cdot (ab)^2 = (ab)^2(ab) \cdot c^2 = ab \cdot c^2 = ac \cdot bc$. Hence S is an RD-AG-groupoid [4, 12, 15]. Conversely, let S be an RD-AG-groupoid and let $a, b, c \in S$. Again by definition of AG-3-band, left invertive law, Lemma 2(ii) and medial law, we have $a \cdot bc = a^2 a \cdot bc = (bc)a \cdot a^2 = (bc)a^2 \cdot a^2 = a^2 a^2 \cdot bc = (a^2 a)a \cdot bc = aa \cdot bc = ab \cdot ac$. Hence S is an LD-AG-groupoid.

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