STABILITY OF LIBRATION POINTS IN THE PHOTOGTAVITATIONAL CIRCULAR RESTRICTED FOUR-BODY PROBLEM WITH THE EFFECT OF PERTURBATIONS AND VARIABLE MASS

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ABSTRACT: *In this paper, we have investigated the stability of the Lagrangian solutions in the photogravitational circular restricted four-body problem with the effect of Coriolis and centrifugal forces, and variable mass. All the three primaries having masses m_1 , m_2 and m_3 form an equilateral triangle, with $m_2 = m_3$ are source of radiation pressures. The infinitesimal body has a mass m which varies with time as per Jeans' Law. The values of the parameters $q = 1/2, k = 0, n = 1$, used by Meshcherskii in the space-time transformations are used. The equations of motion of the infinitesimal body in this current problem differ from the equations of motion of Singh obtained recently in 2015 by the factor of variation of mass parameter γ . The existence of libration points along with their co-linearity character have been examined for a fixed value of parameters γ , α_1 (the proportionality constant in Jeans' law, $0 \le \alpha_1 \le 0.2$), $\mu = 0.019$ (the mass parameter), centrifugal force β and p_i (radiation factors, $0 \le p_i << 1$, i= 1 to 3). It is observed that libration points are depending only on the centrifugal force but the stability depend on both the Coriolis and centrifugal forces. We have also examined the stability of these libration points. We found at most eight libration points in which three are asymptotically stable and the other five are unstable.*

AMS Subject Classification: 70 F 15
Key Words: Circular Restricted four-body problem , Variable mass, Libration points, Perturbations, Coriolis force, centrifugal force, Radiation pressures.

1. INTRODUCTION

Now a day the investigation of the stability of the libration points in the restricted three-body problem with variable mass is one of the most important problems in celestial mechanics. This problem is significant in various fields, particularly to the field of astrodynamics, astronomy and astrophysics. The combined effect of gravitational interaction, perturbation like Coriolis force and centrifugal force and radiation pressure on celestial bodies is an important field of study. Many researchers studied on photogravitational restricted three-body problem like Radzievskii [1,2], Simmons *et al*. [3], Kumar and Choudhary [4], Singh *et al.* [5-8], Mittal *et al.* [9], Papadauris e. al. [10] and Zhang et al. [11] enables to extend the work for the restricted n-body problem.

Bhatnagar *et al*. [12], Singh *et al.* [5, 6, 7, 8], Kalvouridis *et al.* [13], Hallan *et al.* [14,15], Raheem *et al*. [16], Baltagiannis *et al.* [17], Abouelmagd *et al.* [18] etc. have studied in the restricted four body problem with perturbations. The restricted four-body problem has helped in determining the families of simple periodic orbits, nature of the stability around the hyperbolic Lyapunov periodic orbits, stability of the libration points etc. However, the work on the photogravitational circular restricted four-body problem with perturbations and variable mass on such system has been taken as a fresh approach to the invention of n-body problem.

This paper contains various sections considering introduction as first section. In the second section, we have determined the equations of motion and Jacobi integral of the infinitesimal variable mass. In the third section, we have investigated the existence of the libration points for the various fixed parameters. In the fourth section, we have drawn the zero velocity curves. In the fifth section, we have shown the stability of the libration points with the help of the space-time inverse transformations of Meshcherskii [19]. And finally in the sixth section, we have concluded the problem.

2. EQUATIONS OF MOTION

Let there be three masses m_1, m_2 and m_3 ($m_1 \ge m_2 = m_3$). The bodies of masses m_1, m_2 and m_3 are at the vertices of an equilateral triangle of length *l*. They revolve in circular orbits with the angular velocity 'ω' without rotation about their Centre of mass. The angular velocity, ω satisfies the equation

$$\omega^2 l^2 = G(m_1 + m_2 + m_3) \,. \qquad \text{Mccuskey [20]}$$

The line joining the primary m_1 and the mid-point of the line joining the primaries m_2 and m_3 has been chosen as *X*-axis. The Centre of mass of the three primaries has been taken as origin and the line perpendicular to *X*-axis through origin in the plane of motion of the primaries, is taken as *Y*-axis. The line through origin and perpendicular to the plane of motion of the primaries is taken as *Z*-axis. Let us consider a synodic system of coordinates *xyz*; initially coincident with the inertial system *XYZ*, rotating with angular velocity ω about *z*-axis (the z-axis is coincident with Z-axis).

We, now assume that $\dfrac{m_2}{m_1 + m_2 + m_3} = \mu$ and choose units of mass, length and time such that $m_1 + m_2 + m_3 = 1$, $l = 1$ and *G* = 1 respectively. Thus, angular velocity ω =1, masses $m_2 = m_3 = \mu$ and $m_1 = 1-2\mu$. The co-ordinates of the vertices of the equilateral triangle where the masses m_1, m_2 and m_3 are placed, in dimensionless variables are

Table 1 Equilibrium points on $\xi\eta-$plane when μ =0.019, γ=0.4, α_1=0.2, p_1=0.001, p_2=-0.002, p_3=0.003

β	Location of non-collinear libration points (ξ_0, η_0)
1	(**−0.614321, 0.000791**), (−0.632040, −0.378625), (−0.631685, 0.378372), (**−0.127327, 0.607706**), (**−0.125593, -0.608109**), (0.636160, −0.000306), (−0.431382, 0.260404), (−0.431061, −0.260255)
1.2	(**−0.617583, −0.370237**), (−0.617244, 0.370008), (−0.573984, 0.000750), (**−0.139956, 0.566351**), (**−0.138393, −0.566777**), (0.599680, −0.000258), (−0.423074, 0.256097), (−0.422767, −0.255965)
1.3	(**−0.612096, −0.367028**), (−0.611765, 0.366808), (−0.557219, 0.000741), (**−0.145690, 0.548533**), (**−0.144196, −0.548970**), (0.584332, −0.000239), (−0.418644, 0.253889), (−0.418345, −0.253767)
1.4	(**−0.607420, −0.364283**), (−0.607099, 0.364072), (−0.542225, 0.000738), (**−0.151122, 0.532179**), (**−0.149687, −0.532626**), (0.570474, −0.000223), (−0.414057, 0.251678), (−0.413769, −0.251567)
1.44	(**−0.605738, −0.363293**), (−0.605419, 0.363085), (−0.536658, 0.000737), (**−0.153222, 0.525991**), (**−0.151808, −0.526443**), (0.565293, −0.000217), (−0.412184, 0.250799), (−0.411900, −0.250692)

$\left(\sqrt{3}\mu, 0, 0\right)$, $\left(-\sqrt{3}(1-2\mu)/2, -1/2, 0\right)$ and $\left(-\sqrt{3}(1-2\mu)/2, 1/2, 0\right)$ respectively in the synodic axes *xyz*.

Following the procedure of Abouelmagd [18], we can write the equations of motion of the infinitesimal variable mass $m(x, y, z)$ in the photogravitational circular restricted four-body problem with the effect of perturbations in the synodic coordinate system when the variation of the mass is non-isotropic and originates from one point are given by

$$\dot{m}(\dot{x}-\alpha y)+m(\ddot{x}-2\alpha\dot{y})=U_x,$$
$$\dot{m}(\dot{y}+\alpha x)+m(\ddot{y}+2\alpha\dot{x})=U_y,$$
$$\dot{m}\dot{z}+m\ddot{z} \;=U_z, \qquad (1)$$

where

(x, y) = the synodic rectangular coordinates of the infinitesimal mass,

$$U=\frac{m\beta}{2}\left(x^2+y^2\right)+m\left(\frac{(1-2\mu)(1-p_1)}{r_1}+\frac{\mu(1-p_2)}{r_2}+\frac{\mu(1-p_3)}{r_3}\right),$$

$$r_1^{\,2}=\left(x-\sqrt{3}\mu\right)^2+y^2+z^2,$$

$$r_2^{\,2}=\left(x+\frac{\sqrt{3}}{2}(1\text{-}2\mu)\right)^2+\left(y+\frac{1}{2}\right)^2+z^2,$$

$$r_3^{\,2}=\left(x+\frac{\sqrt{3}}{2}(1-2\mu)\right)^2+\left(y-\frac{1}{2}\right)^2+z^2,$$

$\mu=\dfrac{m_2}{m_1+m_2+m_3}$, we have assumed that $m_1 \geq m_2 = m_3$.

The total forces on *m* due to m_i

$=F_i-F_{p_i}=F_i\left(1-\dfrac{F_{p_i}}{F_i}\right)=F_i\left(1-p_i\right), (i=1 to 3)$ towards primaries,

where F_i = gravitational force exerted on *m* due to m_i,

F_{p_i} = solar radiation pressure on *m* due to m_i

$\alpha=1+\varepsilon, |\varepsilon| \ll 1, \ \beta=1+\varepsilon', |\varepsilon'| \ll 1.$

where ε, ε` are small perturbations given to the Coriolis and the centrifugal forces respectively

By Jeans' Law [21]

$$\frac{dm}{dt}=-\alpha_1 m^n,$$

(2)

where α_1 is the constant coefficient and the value of exponent *n* is within the limit $0.4 \leq n \leq 4.4$ for the star of the main sequence. For a rocket $n=1$ and the mass of the rocket which varies exponentially is given by the expression $m=m_0 e^{-\alpha_1 t}$, $m=m_0$ at $t=0$.

To simplify the equations of motion, we use the space-time transformations

$$\xi=\gamma^q x, \eta=\gamma^q y, \zeta=\gamma^q z, d\Gamma=\gamma^k dt,$$
$$r_1=\gamma^{-q}\rho_1, \ r_2=\gamma^{-q}\rho_2, r_3=\gamma^{-q}\rho_3, \qquad (3)$$

where $\gamma=\dfrac{m}{m_0}$. (4)

Now, taking $q=1/2, k=0, n=1,$ using (2), (3) and (4), the equations of motion (1) become

$$\ddot{\xi}-2\alpha\dot{\eta}=W_\xi,$$
$$\ddot{\eta}+2\alpha\dot{\xi}=W_\eta,$$
$$\ddot{\zeta} \qquad =W_\zeta, \qquad (5)$$

where

$$W=\frac{1}{2}\left(\beta+\frac{\alpha_1^2}{4}\right)\left(\xi^2+\eta^2\right)+\frac{\alpha_1^2}{8}\zeta^2 \;,$$
$$+\gamma^{3/2}\left(\frac{(1-2\mu)(1-p_1)}{\rho_1}+\frac{\mu(1-p_2)}{\rho_2}+\frac{\mu(1-p_3)}{\rho_3}\right)$$

$$\rho_1^{\,2}=\left(\xi-\sqrt{3}\,\mu\gamma^{1/2}\right)^2+\eta^2+\zeta^2,$$

$$\rho_2^{\,2}=\left(\xi+\frac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)^2+\left(\eta\ +\frac{1}{2}\gamma^{1/2}\right)^2+\zeta^2,$$

$$\rho_3^{\,2}=\left(\xi+\frac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)^2+\left(\eta-\frac{1}{2}\gamma^{1/2}\right)^2+\zeta^2.$$

From equations (5), we can write that

$\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2=2W-C,$ where C is the Jacobi integral constant.

3. LIBRATION POINTS

The libration points with variable mass m are obtained from solution of the equations:

$W_\xi=0,\ W_\eta=0,\ W_\zeta=0$ i.e.

$$\left(\beta+\frac{\alpha_1^2}{4}\right)\xi-\gamma^{3/2}\left[\frac{(1-2\mu)\ (1-p_1)\left(\xi-\sqrt{3}\ \mu\gamma^{1/2}\right)}{{\rho_1}^3}+\frac{\mu(1-p_2)\left(\xi\ +\ \frac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)}{{\rho_2}^3}+\frac{\mu(1-p_3)\left(\xi+\frac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)}{{\rho_3}^3}\right]=0,$$

Table 2 Equilibrium points on $\xi\eta-\text{plane}$ When μ =0.019, γ=0.4, α_1=0.2, p_1=0.001, p_2=-0.002, p_3=0.003

β	Location of non-collinear libration points (ξ_0,η_0)
1	(−0.633233, −0.379243), (−0.633198, 0.379218), (−0.616542, 0.000077), (**−0.122713, 0.610756**), (**−0.122541, −0.610795**), (0.638075, −0.000030), (−0.431218, 0.260258), (−0.431187, −0.260243)
1.2	(**−0.618592, −0.370780**), (−0.618558, 0.370758), (−0.575994, 0.000073), (**−0.135740, 0.569381**), (**−0.135586, −0.569422**), (0.601480, −0.000025), (−0.423059, 0.256010), (−0.423029, −0.255996)
1.3	(**−0.613034, −0.367539**), (−0.613001, 0.367517), (−0.559138, 0.000072), (**−0.141627, 0.551563**), (**−0.141479, −0.551605**), (0.586084, −0.000024), (−0.418710, 0.253829), (−0.418681, −0.253817)
1.4	(**−0.608299, −0.364767**), (−0.608267, 0.364746), (−0.544061, 0.000072), (**−0.147187, 0.535214**), (**−0.147045, −0.535257**), (0.572183, −0.000022), (−0.414208, 0.251642), (−0.414180, −0.251631)
1.44	(**−0.606594, −0.363766**), (−0.606563, 0.363746), (−0.538464, 0.000072), (**−0.149332, 0.529030**), (**−0.149192, −0.529073**), (0.566985, −0.000021), (−0.412370, 0.250771), (−0.412342, −0.250760).

$$\left(\beta+\frac{\alpha_1^2}{4}\right)\eta-\gamma^{3/2}\left(\frac{(1-2\mu)(1-p_1)\eta}{{\rho_1}^3}+\frac{\mu(1-p_2)\left(\eta+\frac{1}{2}\gamma^{1/2}\right)}{{\rho_2}^3}+\frac{\mu(1-p_3)\left(\eta-\frac{1}{2}\gamma^{1/2}\right)}{{\rho_3}^3}\right)=0,$$

$$\frac{\alpha_1^2}{4}\zeta-\gamma^{3/2}\left(\frac{(1-2\mu)(1-p_1)\zeta}{{\rho_1}^3}+\frac{\mu(1-p_2)\zeta}{{\rho_2}^3}+\frac{\mu(1-p_3)\zeta}{{\rho_3}^3}\right)=0\cdot \quad (6)$$

If we take $\alpha_1=0$ or $\gamma=1$ and $p_i=0,(\ i=1\,\text{to}\,3)$ in equations (6), we obtain the equations of Singh *et al.* [8].

For $\alpha_1\neq 0$ and $p_i\neq 0(\ i=1\,to\,3)$, we have determined the positions of libration points numerically for different values of α_1, γ and p_i in the plane of motion of the primaries i.e. in $\xi\eta-\text{plane}$ (Tables 1 to 7). Besides, we have determined the locations of the libration points in the out-of-plane (Table 7).

Table 3 Equilibrium points on $\xi\eta-\text{plane}$ when

$\gamma=0.9, \alpha_1=0.2,\ p_1=0.01,\ p_2=0.02,\ p_3=0.03.$

β	Location of non-collinear libration points (ξ_0,η_0)
1	(**−0.948061, −0.567937**), (−0.947528, 0.567558), (−0.921482, 0.001186), (**−0.190990, 0.911559**), (**−0.188390, −0.912164**), 0.954241, −0.000460), (−0.647073, 0.390607), (−0.646592, −0.390383)
1.2	(**−0.926375, −0.555356**), (−0.925866, 0.555012), (−0.860977, 0.001125), (**−0.209934, 0.849526**), (**−0.207590, −0.850165**), (0.899520, −0.000387), (−0.634611, 0.384146), (−0.634150, −0.383948)
1.3	(**−0.918144, −0.550542**), (−0.917648, 0.550212), (−0.835829, 0.001112), (**−0.218536, 0.822800**), (**−0.216294, −0.823456**), (0.876498, −0.000359), (−0.627966, 0.380834), (−0.627518, −0.380651)
1.4	(**−0.911131, −0.546425**), (−0.910648, 0.546108), (−0.813338, 0.001107), (**−0.226684, 0.798268**), (**−0.224531, −0.798940**), (0.855711, −0.000335), (−0.621086, 0.377517), (−0.620654, −0.377351)
1.44	(**−0.908607, −0.544940**), (−0.908129, 0.544628), (−0.804988, 0.001106), (**−0.229833, 0.788987**), (**−0.227713, −0.789665**), (0.847939, −0.000326), (−0.618277, 0.376199), (−0.617851, −0.376039)

Table 4 Equilibrium points on $\xi\eta-\text{plane}$ when

$\gamma=0.9,\ \alpha_1=0.2,\ p_1=0.001,\ p_2=0.002,\ p_3=0.003.$

β	Location of non-collinear libration points (ξ_0,η_0)
1	(**−1.002647, −0.600449**), (−1.002591, 0.600410), (−0.978504, 0.000123), (**−0.192912, 0.969355**), (**−0.192639, −0.9694161**), (1.012174, −0.000049), (−0.682428, 0.411830), (−0.682378, −0.411807)

1.2	(**−0.979044, −0.586816**), (−0.978991, 0.586780), (−0.913568, 0.000116), (**−0.213662, 0.903233**), (**−0.213416, −0.903297**), (0.953606, −0.000041), (−0.669589, 0.405130), (−0.669541, −0.405109),
1.3	(**−0.970107, −0.581607**), (−0.970055, 0.581572), (−0.886605, 0.000114), (**−0.223026, 0.874804**), (**−0.222791, −0.874870**), (0.929002, −0.000038), (−0.662739, 0.401684), (−0.662692, −0.401665)
1.4	(**−0.962504, −0.577157**), (−0.962453, 0.577124), (−0.862507, 0.000114), (**−0.231865, 0.848740**), (**−0.231640, −0.848808**), (0.906803, −0.000035), (−0.655643, 0.398226), (−0.655598, −0.398208)
1.44	(**−0.959770, −0.575553**), −0.959720, 0.575520), (−0.853565, 0.000114), (**−0.235274, 0.838887**), (**−0.235052, −0.838955**), (0.898509, −0.000034), (−0.652744, 0.396847), (−0.652700, −0.396831),

Table 5 Equilibrium points on $\xi\eta-$plane when

$\gamma=1, \alpha_1=0, p_1=0.001, p_2=0.002, p_3=0.003$ (Corresponding to the classical case of restricted four-body problem)

β	Location of non-collinear libration points (ξ_0, η_0)
1	(**−0.949850, −0.568865**), (−0.949797, 0.568828), (−0.924814, 0.000116), (**−0.184070, 0.916134**), (**−0.183812, −0.916192**), (0.957112, −0.000046), (−0.646827, 0.390387), (−0.646780, −0.390365)
1.2	(**−0.927888, −0.556171**), (−0.927838, 0.556137), (−0.863992, 0.000110), (**−0.203611, 0.854071**), (**−0.203379, −0.854133**), (0.902220, −0.0000388), (−0.634589, 0.384015), (−0.634544, −0.383995)
1.3	(**−0.919551, −0.551309**), (−0.919502, 0.551276), (−0.838707, 0.000108), (**−0.212441, 0.827345**), (**−0.212219, −0.827408**), (0.879127, −0.000036), (−0.628066, 0.380743), (−0.628022, −0.380725)
1.4	(**−0.912448, −0.547150**), (−0.912401, 0.547119), (−0.816092, 0.000108), (**−0.220781, 0.802821**), (**−0.220568, −0.802886**), (0.858274, −0.0000335), (−0.621313, 0.377463), (−0.621270, −0.377446)
1.44	(**−0.909892, −0.545650**), (−0.909845, 0.545619), (−0.807696, 0.0001083), (**−0.223998, 0.793545**), (**−0.223789, −0.793610**), (0.850478, −0.0000326), (−0.618555, 0.376157), (−0.618513, −0.376141)

Table 6 Equilibrium points on $\xi\eta-$plane when

$\gamma=1, \alpha_1=0, p_1=0.01, p_2=0.02, p_3=0.03,$

(Corresponding to the classical case of restricted four-body problem)

β	Location of non-collinear libration points (ξ_0, η_0)
1	(**−1.000743, −0.599465**), (−1.000181, 0.599063), (−0.974974, 0.001256), (**−0.200245, 0.964530**), (**−0.197489, −0.965166**), (1.009137, −0.000489), (−0.682698, 0.412066), (−0.682191, −0.411830)
1.2	(**−0.977437, −0.585952**), (−0.976898, 0.585588), (−0.910374, 0.001188), (**−0.220353, 0.898441**), (**−0.217871, −0.899113**), (0.950751, −0.000412), (−0.669624, 0.405273), (−0.669138, −0.405063),
1.3	(**−0.968613, −0.5807941**), (−0.968089, 0.580445), (−0.883558, 0.001173), (**−0.229473, 0.870013**), (**−0.227100, −0.870703**), (0.926224, −0.000381), (−0.662646, 0.401784), (−0.662173, −0.401590),
1.4	(**−0.961106, −0.576388**), (−0.960596, 0.576053), (−0.859592, 0.001167), (**−0.238106, 0.843942**), (**−0.235828, −0.844648**), (0.904095, −0.000355), (−0.655417, 0.398287), (−0.654960, −0.398109)
1.44	(**−0.958406, −0.574800**), (−0.957902, 0.574470), (−0.850698, 0.001166), (**−0.241441, 0.834083**), (**−0.239198, −0.834796**), (0.895826, −0.000346), (−0.652464, 0.396895), (−0.652013, −0.396725)

Table 7 Out-of-plane equilibrium points when

β	Location of libration points (ξ_0, ζ_0)
1	(−0.0000172, ± 3.267847), (−0.686833, 0), (0.711249, 0)
1.2	(−0.0000143, ±3.267847), (−0.641735, 0), (0.670463, 0)
1.3	(−0.000013, ±3.267847), (−0.622991, 0), (0.653303, 0)
1.4	(−0.0000123, ±3.267847), (−0.606227, 0), (0.637809, 0)
1.44	(−0.0000119, ±3.267847), (−0.6000032, 0), (0.632017, 0)

$\gamma=0.5, \alpha_1=0.2, \eta=0, p_1=0.01, p_2=0.02, p_3=0.03,$

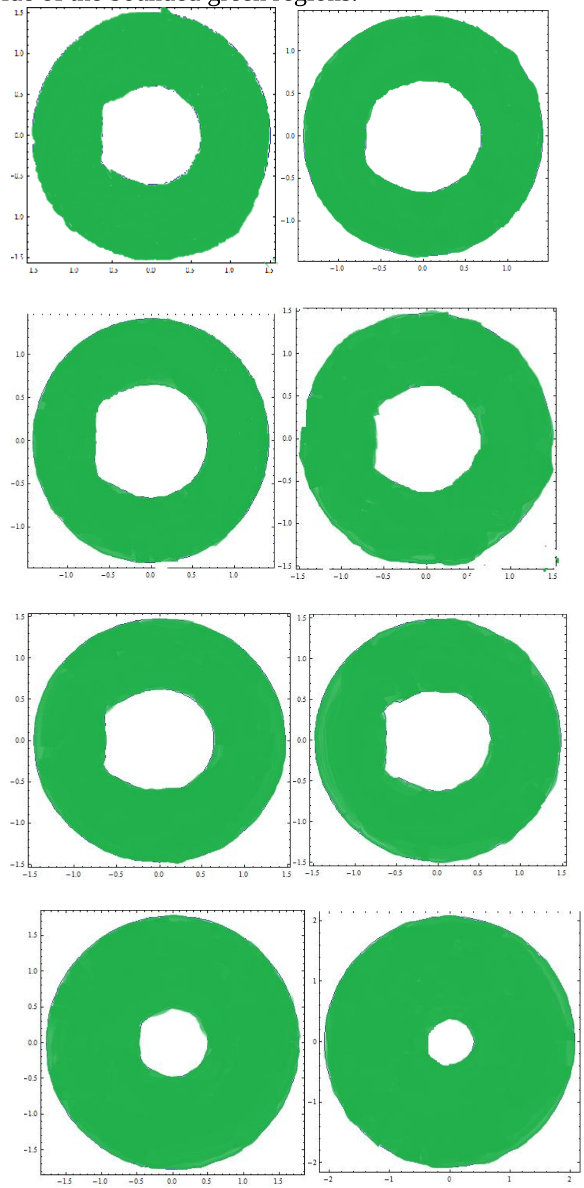
The Tables 1 to 6 shows that there is no collinear libration point and at most eight non-collinear libration points exist. The table 7 shows that there are two non-collinear libration points which are symmetrical about $\xi-$axis and two collinear libration points. For $\alpha_1>0$, the fourth infinitesimal mass decreases with time and the libration points continuously approach towards the origin of the coordinate system $\xi\eta\zeta$.

4. ZERO VELOCITY CURVE

Zero velocity curves define boundary, which separates regions where the motion is allowed or forbidden. We can draw the zero velocity curves by the relation 2W- C = 0. Thus, the equation

$$\left(\beta+\frac{\alpha_1^2}{4}\right)\left(\xi^2+\eta^2\right)+2\gamma^{3/2}\left(\frac{(1-2\mu)(1-p_1)}{\rho_1}+\frac{\mu(1-p_2)}{\rho_2}+\frac{\mu(1-p_3)}{\rho_3}\right)+\frac{\alpha_1^2}{4}\zeta^2=C$$

gives more information about the possible dynamics at a Jacobi constant C. We have drawn the zero velocity curves of our problem for different values of the Jacobian constant C corresponding to the table 1 for the value β = 1 and similarly we can draw the zero velocity curves for all the values of β. The infinitesimal body is free to move only outside of the bounded green regions.



Zero velocity curves for different values of C corresponding to the libration points for $\beta=1$, in the table-1.

5 STABILITY OF THE LIBRATION POINTS

For the linear stability of the libration points, we give displacement to (ξ_0,η_0,ζ_0) as

$$\xi=\xi_0+u\,,\ \eta=\eta_0+v,\ \zeta=\zeta_0+w,$$
$$(u,v,w)<<1 \tag{7}$$

where (ξ_0,η_0,ζ_0) is the libration point for a fixed value of time t.

From the equations (5) and (7), we obtain the variational equations as

$$\ddot{u}-2\alpha\dot{v}=(W_{\xi\xi})_0u+(W_{\xi\eta})_0v+(W_{\xi\zeta})_0w,$$
$$\ddot{v}+2\alpha\dot{u}=(W_{\eta\xi})_0u+(W_{\eta\eta})_0v+(W_{\eta\zeta})_0w,$$
$$\ddot{w}=(W_{\zeta\xi})_0u+(W_{\zeta\eta})_0v+(W_{\zeta\zeta})_0w, \tag{8}$$

where the subscript '0' in equations (8) represents that the values are to be found at the libration point (ξ_0,η_0,ζ_0) under consideration. We have

$$W_{\xi\xi}=\left(\beta+\frac{\alpha_1^2}{4}\right)+3\gamma^{3/2}\left\{(1-2\mu)\frac{(1-p_1)\left(\xi-\sqrt{3}\,\mu\gamma^{1/2}\right)^2}{\rho_1^{5}}+\mu\frac{(1-p_2)\left(\xi+\frac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)^2}{\rho_2^{5}}+\mu\frac{(1-p_3)\left(\xi+\frac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)^2}{\rho_3^{5}}\right\}-\gamma^{3/2}\left(\frac{(1-2\mu)(1-p_1)}{\rho_1^{3}}+\frac{\mu(1-p_2)}{\rho_2^{3}}+\frac{\mu(1-p_3)}{\rho_3^{3}}\right),$$

$$W_{\eta\xi}=W_{\xi\eta}=3\gamma^{3/2}\left\{(1-2\mu)\frac{(1-p_1)\left(\xi-\sqrt{3}\mu\gamma^{1/2}\right)\eta}{\rho_1^{5}}+\mu\frac{(1-p_2)\left(\xi+\frac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)\left(\eta+\frac{\gamma^{1/2}}{2}\right)}{\rho_2^{5}}+\mu\frac{(1-p_3)\left(\xi+\frac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)\left(\eta-\frac{\gamma^{1/2}}{2}\right)}{\rho_3^{5}}\right\},$$

$$W_{\eta\eta}=\left(\beta+\frac{\alpha_1^2}{4}\right)+3\gamma^{3/2}\left\{(1-2\mu)\frac{(1-p_1)\eta^2}{\rho_1^{5}}+\mu\frac{(1-p_2)\left(\eta+\frac{\gamma^{1/2}}{2}\right)^2}{\rho_2^{5}}+\mu\frac{(1-p_3)\left(\eta-\frac{\gamma^{1/2}}{2}\right)^2}{\rho_3^{5}}\right\}-\gamma^{3/2}\left(\frac{(1-2\mu)(1-p_1)}{\rho_1^{3}}+\frac{\mu(1-p_2)}{\rho_2^{3}}+\frac{\mu(1-p_3)}{\rho_3^{3}}\right),$$

$$W_{\xi\zeta}=W_{\zeta\xi}=3\gamma^{3/2}\zeta\left\{\begin{array}{l}(1-2\mu)\dfrac{(1-p_1)\left(\xi-\sqrt{3}\mu\gamma^{1/2}\right)}{\rho_1^{\,5}}\\ +\mu\dfrac{(1-p_2)\left(\xi+\dfrac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)}{\rho_2^{\,5}}\\ +\mu\dfrac{(1-p_3)\left(\xi+\dfrac{\sqrt{3}}{2}(1-2\mu)\gamma^{1/2}\right)}{\rho_3^{\,5}}\end{array}\right\},$$

$$W_{\eta\zeta}=W_{\zeta\eta}=3\gamma^{3/2}\zeta\left\{\begin{array}{l}(1-2\mu)\dfrac{(1-p_1)\eta}{\rho_1^{\,5}}\\ +\mu\dfrac{(1-p_2)\left(\eta+\dfrac{\gamma^{1/2}}{2}\right)}{\rho_2^{\,5}}\\ +\mu\dfrac{(1-p_3)\left(\eta-\dfrac{\gamma^{1/2}}{2}\right)}{\rho_3^{\,5}}\end{array}\right\}$$

$$W_{\zeta\zeta}=\frac{\alpha_1^2}{4}+3\gamma^{3/2}\zeta^2\left(\begin{array}{l}(1-2\mu)\dfrac{(1-p_1)}{\rho_1^{\,5}}\\ +\mu\dfrac{(1-p_2)}{\rho_2^{\,5}}+\mu\dfrac{(1-p_3)}{\rho_3^{\,5}}\end{array}\right)-\gamma^{3/2}\left(\begin{array}{l}\dfrac{(1-2\mu)(1-p_1)}{\rho_1^{\,3}}\\ +\dfrac{\mu(1-p_2)}{\rho_2^{\,3}}+\dfrac{\mu(1-p_3)}{\rho_3^{\,3}}\end{array}\right), \qquad (9)$$

If we take $\alpha_1=0$, the system (8) reduced to a system with constant mass. For $\alpha_1>0$, the coordinates of the three primaries vary with time *t* and their distances to the libration point (ξ_0,η_0,ζ_0) decrease with time. So, we use the space-time inverse transformations of Meshcherskii [19] i.e. $x=\gamma^{-1/2}\xi$, $y=\gamma^{-1/2}\eta$, $z=\gamma^{-1/2}\zeta$. The positions of the primaries are fixed therefore their distances to the libration points will not be changed.

In phase-space, equations (8) may be written as

$$\dot{u}=u_1,\qquad \dot{v}=v_1,\qquad \dot{w}=w_1,$$
$$\dot{u}_1-2\alpha\, v_1=(W_{\xi\xi})_0u+(W_{\xi\eta})_0v+(W_{\xi\zeta})_0w,$$
$$\dot{v}_1+2\alpha\, u_1=(W_{\eta\xi})_0u+(W_{\eta\eta})_0v+(W_{\eta\zeta})_0w,$$
$$\dot{w}_1\qquad=(W_{\zeta\xi})_0u+(W_{\zeta\eta})_0v+(W_{\zeta\zeta})_0w. \qquad (10)$$

Using Meshcherskii [19] inverse transformations, and putting

$$x'=\gamma^{-1/2}u,\ y'=\gamma^{-1/2}v,\ z'=\gamma^{-1/2}w,$$
$$u'=\gamma^{-1/2}u_1,\qquad v'=\gamma^{-1/2}v_1,\qquad w'=\gamma^{-1/2}w_1,$$

in the matrix, the system (10) can be written as follows:

$$\begin{pmatrix}\frac{dx'}{dt}\\ \frac{dy'}{dt}\\ \frac{dz'}{dt}\\ \frac{du'}{dt}\\ \frac{dv'}{dt}\\ \frac{dw'}{dt}\end{pmatrix}=\begin{pmatrix}\frac{\alpha_1}{2}&0&0&1&0&0\\ 0&\frac{\alpha_1}{2}&0&0&1&0\\ 0&0&\frac{\alpha_1}{2}&0&0&1\\ (W_{\xi\xi})_0&(W_{\xi\eta})_0&0&\frac{\alpha_1}{2}&2\alpha&0\\ (W_{\eta\xi})_0&(W_{\eta\eta})_0&0&-2\alpha&\frac{\alpha_1}{2}&0\\ 0&0&(W_{\zeta\zeta})_0&0&0&\frac{\alpha_1}{2}\end{pmatrix}\times\begin{pmatrix}x'\\ y'\\ z'\\ u'\\ v'\\ w'\end{pmatrix}. \qquad (11)$$

The linear stability of the solution of the matrix depends on the existence of stable region of the libration`n point, which in turn depends on the boundedness of the solution of linear and homogenous system of equations (11). We have determined the linear stability of the libration points. For this, we found the characteristic roots of the coefficient matrix of equation (11) numerically.

The characteristic equation of the coefficient matrix is

$$\lambda^6-3\alpha_1\lambda^5+\left(\frac{15}{4}\alpha_1^2+P\right)\lambda^4-\left(\frac{5}{2}\alpha_1^3+2P\alpha_1\right)\lambda^3+\left(\frac{15}{16}\alpha_1^4+\frac{3}{2}P\alpha_1^2+Q\right)\lambda^2$$
$$-\left(\frac{3}{16}\alpha_1^5+\frac{1}{2}P\alpha_1^3+Q\alpha_1\right)\lambda+\left(\frac{1}{64}\alpha_1^6+\frac{1}{16}P\alpha_1^4+\frac{1}{4}Q\alpha_1^2+R\right)=0, \qquad (12)$$

where

$$P=4\alpha^2-(W_{\xi\xi})_0-(W_{\eta\eta})_0-(W_{\zeta\zeta})_0,$$

$$Q=(W_{\xi\xi})_0(W_{\zeta\zeta})_0+(W_{\eta\eta})_0(W_{\zeta\zeta})_0+(W_{\xi\xi})_0(W_{\eta\eta})_0$$
$$-4\alpha^2(W_{\zeta\zeta})_0-(W_{\xi\eta})_0^2,$$
$$R=-(W_{\xi\xi})_0(W_{\eta\eta})_0(W_{\zeta\zeta})_0+(W_{\zeta\zeta})_0(W_{\xi\eta})_0^2,$$

where values of $W_{\xi\xi}, W_{\eta\eta}, W_{\zeta\zeta}$ and $W_{\xi\eta}$ are given by equations (9).

The characteristic roots of the equation (12) have been calculated at various libration points in the range $0<\gamma\le 1,\ 0\le\alpha_1\le 0.2,\ \mu=0.019, 1\le\beta\le 1.44$ and $p_i \ll 1\ (i=1\text{to}\,3)$.

We have observed that in all eight liberation points, for three liberation points (dark black coordinates in the tables) there exists complex characteristic roots, means the liberations points are asymptotically stable and rest five liberation points have at least one real characteristic root, means that these five libration points are unstable.

5 CONCLUSION AND DISSCUSSION

We have investigated existence and stability of the libration points in the photogravitational circular restricted four-body problem with the effect of perturbation and variable mass and the primaries with masses m_1 , m_2 *and* m_3 ($m_1\ge m_2=m_3$). It has been found that at most eight libration points exist and all are non-collinear. We have also found that the libration points exist in the out-of-plane (Table 7). Also we have drawn the zero velocity curves and found that the infinitesimal mass will move only in the particular region of the whole region and forbidden in the bounded green region. And finally, we have investigated the stability of the liberation points. And found that out of eight liberation points, three are asymptotically stable (dark black in the tables) and five are unstable for all values of the parameters. Our result is differ from the result of Singh et al. [8] as they found that two out of eight liberation points are stable and rest six are unstable.

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REFERENCES

1. Radzievskii, V.V. The Restricted Problem of Three Bodies Taking Account of Light Pressure. Astronomical Journal, 27, 249 (1950).
2. Radzievskii, V.V. The Space Photogravitational Restricted Three-Body Problem. Astronomical Journal, 30, 225 (1953).
3. Simmons, JFL., Mcdonald, AJC. and Brown, JC The restricted three body problem with radiation pressure. Celest. Mech. 35, 145 (1985).
4. Kumar, V. and Chaudhary, R.K. Existence of libration points in the generalized photogravitational restricted problem of three bodies. Celest. Mech. 39, 159-171 (1986).
5. Singh, J., Ishwar, B. Effect of perturbations on the location of equilibrium points in the restricted problem of three bodies with variable mass. Celest. Mech. 32(4), 297-30 (1984).
6. Singh, J., Ishwar, B. Effect of perturbations on the stability of triangular points in the restricted problem of three bodies with variable mass. Celest. Mech. 35, 201-207 (1985).
7. Singh, J., Leke, Stability of photogravitational restricted three-body problem with variable mass, Astrophysics and Space Sci. 326 (2), 305-314 (2010).
8. Singh, J., Vincent, A. E. Effect of perturbations in the Coriolis and centrifugal forces on the stability of equilibrium points in the restricted four-body problem. Few-Body Syst. 56**,** 713–723 (2015), DOI 10.1007/s00601-015-1019-3.
9. Mittal, A., Ahmad, I. and Bhatnagar, K.B. Periodic orbits in the photogravitational restricted problem with the smaller primary an oblate body. Astrophys Space Sci 323, 65-73 (2009).
10. Papadouris, J.P., Papadakis, K.E. Equilibrium points in the photogravitational restricted four-body problem. Astrophysics and Space Science. 344, 21–38 (2013), DOI 10.1007/s10509-012-1319-8.
11. Zhang, M. J., Zhao, C. Y., Xiong, Y. Q. On the triangular libration points in photogravitational restricted three-body problem with variable mass. Astrophysics Space Sci. 337, 107-113 (2012), DOI 10.1007/s10509-011-0821-8.
12. Bhatnagar, K.B., Hallan, P.P. Effect of perturbations in Coriolis and centrifugal forces on the stability of libration points in the restricted problem. Celest. Mech. 18, 105-112 (1978).
13. Kalvouridis, T.J., Arribas, M., Elipe, A. Parametric evolution of periodic orbits in the restricted four-body problem with radiation pressure. Planet. Space Sci. 55, 475-493 (2007), DOI 10.1016/j.pss.2006.07.005.
14. Hallan, P. P. and Rana, N. Effect of perturbations in coriolis and centrifugal forces on the location and stability of the equilibrium point in the Robe's circular restricted three body problem. Planetary and Space Science, 49(9), 957–960 (2001).
15. Hallan, P. P. and Mangang, K. B. Effect of perturbations in coriolis and centrifugal forces on the Nonlinear Stability of Equilibrium point in Robe`s Restricted Circular Three body Problem. Advances in Astronomy, 21 (2008).
16. Raheem, A., Singh, J. Combined effects of perturbations, radiations and oblateness on the stability of equilibrium points in the restricted three body problem. Astronomical Journal 131(3), 1880-188 (2006).
17. Baltagiannis, A.N., Papadakis, K.E. Equilibrium points and their stability in the restricted four-body problem. International Journal of Bifurcation and Chaos 21, 2179 (2011), DOI10.1142/S0218127411029707.
18. Abouelmagd, E.I., Mostafa, A. Out of plane equilibrium points locations and the forbidden movement regions in the restricted three-body problem with variable mass. Astrophysics and Space Sci. 357, 58 (2015), DOI 10.1007/s10509-015-2294-7.
19. Meshcherskii, I.V. Studies on the mechanics of bodies of variable mass. GITTL, Moscow (1949).
20. Mccuskey, S.W. Introduction to celestial mechanics, Addison-Wesley Publishing Company, Inc. USA (1963).
21. Jeans, J. H. Astronomy and Cosmogony. Cambridge University Press, Cambridge (1928).