

ON CHROMATICITY OF LADDER-TYPE GRAPHS

Abdul Rauf Nizami, Mobeen Munir, Amjad Shahbaz Khan, Zakia Shahzadi

Division of Science and Technology, University of Education, Lahore-Pakistan

E-mail: arnizami@ue.edu.pk, mmunir@ue.edu.pk, amjadshahbaz7t@gmail.com, zakiashahzadi@yahoo.com

ABSTRACT: We give general formulas of chromatic polynomial of some interesting families of Ladder-Type graphs, and conclude that, except two, neither two of them are chromatically equivalent. Moreover, some of them are not chromatically unique.

Mathematics Subject Classification: Primary 05C31; Secondary 57M27

Keywords: Chromatic polynomial, Chromatic equivalence, Chromatic uniqueness, λ -coloring, Ladder-type graph

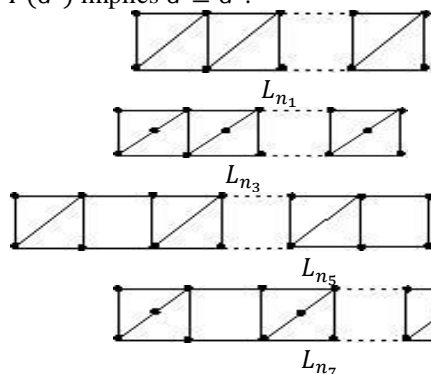
1. INTRODUCTION

The chromatic polynomial was introduced by G. D. Birkhoff in 1912 as a function that counts the number of graph colorings for planer graphs to solve the four color problem [1]. In 1932 H. Whitney generalized it from the planer graphs to the arbitrary graphs [7]. The chromatic polynomial, because of its theoretical and applied importance, has generated a large body of work. Chia [4] provides an extensive bibliography on the chromatic polynomial, and Dong, Koh, and Teo [6] gave a comprehensive treatment. The following two operations are essential to understand the chromatic polynomial definition for a graph G . These are *edge deletion*, denoted by $G'' = G - e$, and *edge contraction*, denoted by $G'' = G/e$.

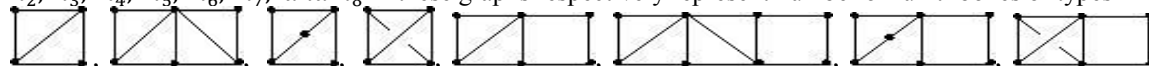
Definition 1.1. The chromatic polynomial is a function P from the set of graphs to the set $\mathbb{Z}[\lambda]$, a ring of polynomials, such that

$$P(G) = \begin{cases} 0 & \text{if there is a loop in } G \\ \lambda^n & \text{if } G \text{ consists of only } n \text{ isolated vertices} \\ P(G - e) - P(G/e) & \text{otherwise} \end{cases}$$

Two graphs are chromatically equivalent if they have the same chromatic polynomial; a graph G is chromatically unique if $P(G) = P(G')$ implies $G \cong G'$.



The subscript $n_1, n_2, n_3, n_4, n_5, n_6, n_7$, and n_8 in these graphs respectively represent number of 'unit' boxes of types



The following is the chromatic polynomial of the ladder graph L_n , which already exists in the literature.

Proposition 1.4. The chromatic polynomial of the graph L_n , is

$$P(L_n) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^n.$$

First we give the chromatic polynomials of four 'basic' ladder-type graphs:

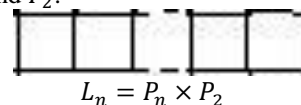
Theorem 1.5. The chromatic polynomials of L_{n1} , L_{n2} , L_{n3} , and L_{n4} are

$$a) P(L_{n1}) = \lambda(\lambda - 1)(\lambda - 2)^{2n_1},$$

For a positive integer λ , a λ -coloring of a graph G is a mapping of $V(G)$ into the set $\{1, 2, 3, \dots, \lambda\}$ of λ colors. Thus, there are exactly λ^n colorings for a graph on n vertices. If φ is a λ -coloring such that $\varphi(u) \neq \varphi(v)$ for all $uv \in E$, then φ is called a proper (or admissible) coloring. The chromatic number of a graph G , denoted by $\gamma(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

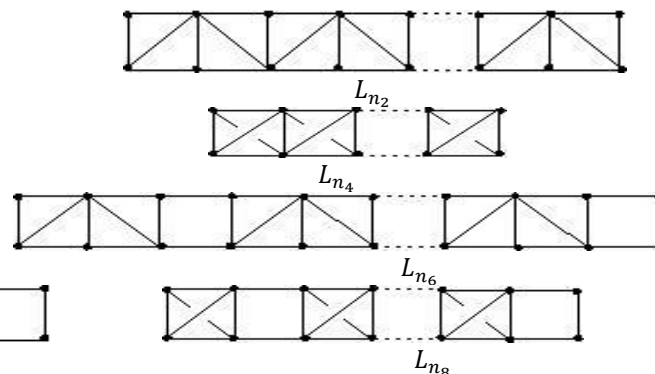
Remark 1.2. Every evaluation of chromatic polynomial at some number λ actually gives the λ -coloring of the graph. Since we are interested mainly in ladder-type graphs, we define them here. First, the two closely related definitions:

Definition 1.3. A ladder graph L_n is the Cartesian product of path graphs P_n and P_2 :



We define a *ladder-type graph* a ladder graph with addition of some edges and vertices, in some pattern, keeping the main structure of L_n intact.

The ladder type graphs we are concerned with are:



$$b) P(L_{n2}) = \lambda(\lambda - 1)(\lambda - 2)^{4n_2},$$

$$c) P(L_{n3}) = \lambda(\lambda - 1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7)^{n_3},$$

$$d) P(L_{n4}) = \lambda(\lambda - 1)(\lambda - 2)^{n_4}(\lambda - 3)^{n_4}$$

Then we have the proposition:

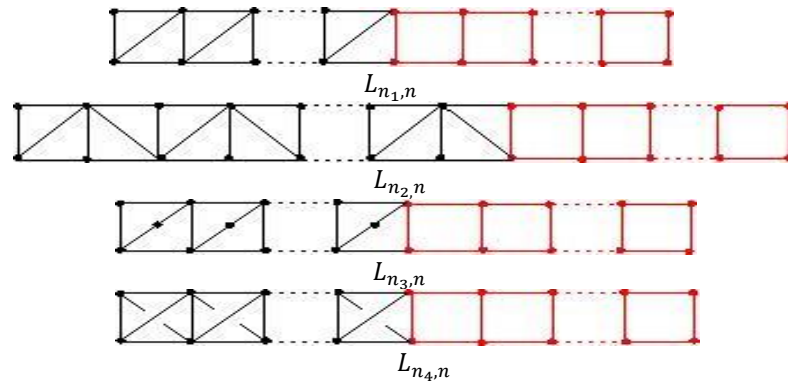
Proposition 1.6. The chromatic polynomials of L_{n5} , L_{n6} , L_{n7} , and L_{n8} are

$$a) P(L_{n5}) = \lambda(\lambda - 1)(\lambda - 2)^{2n_5}(\lambda^2 - 3\lambda + 3)^{n_5}$$

$$b) P(L_{n6}) = \lambda(\lambda - 1)(\lambda - 2)^{2n_6}(\lambda^2 - 3\lambda + 3)^{n_6}$$

- c) $P(L_{n_7}) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)^{n_7}(\lambda^3 - 5\lambda^2 + 10\lambda - 7)^{n_7}$, and
 d) $P(L_{n_6}) = \lambda(\lambda - 1)(\lambda - 2)^{n_8}(\lambda - 3)^{n_8}(\lambda^2 - 3\lambda + 3)^{n_8}$

Besides the above graphs, the following are special types of ladder-type graphs. These are actually obtained by appending the ladder graph L_n to the graphs $L_{n_1}, L_{n_2}, L_{n_3}$, and L_{n_4} .



We shall give the chromatic polynomials of these graphs as a corollary of the general result:

Theorem 1.7. If a graph G is obtained by appending L_n to a graph G_1 such that they share nothing except just one edge, then

$$P(G) = (\lambda^2 - 3\lambda + 3)^n P(G_1).$$

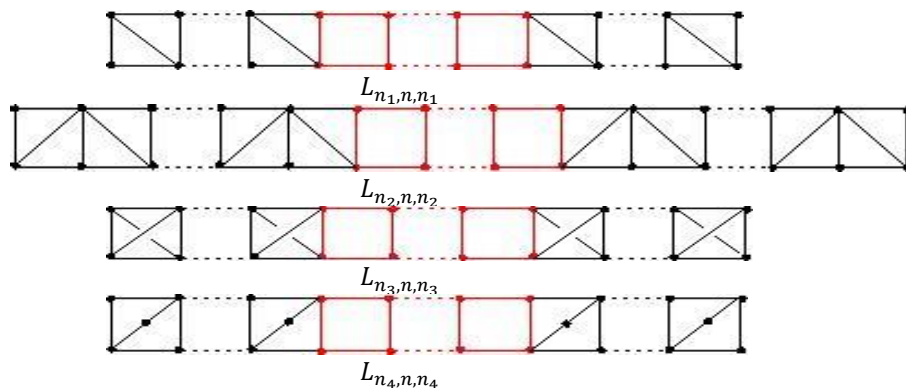
Corollary 1.8.

- a) $P(L_{n_1,n}) = \lambda(\lambda - 1)(\lambda - 2)^{2n_1}$
 b) $P(L_{n_2,n}) = \lambda(\lambda - 1)(\lambda - 2)^{4n_2}[\lambda^2 - 3\lambda + 3]^n$.

c) $P(L_{n_3,n}) = \lambda(\lambda - 1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7)^{n_3}[\lambda^2 - 3\lambda + 3]^n$

d) $P(L_{n_4,n}) = \lambda(\lambda - 1)(\lambda - 2)^{n_4}(\lambda - 3)^{n_4}[\lambda^2 - 3\lambda + 3]^n$

If L_n is sandwiched between ladder-type graph L_{n_i} , $1 \leq i \leq 4$, then we shall denote the resultant ladder-type graph by L_{n_i,n,n_i} . The chromatic polynomial of the following graphs are given in a lemma:



Lemma 1.9.

- a) $P(L_{n_1,n,n_1}) = (\lambda - 2)^{2n_1} P(L_{n_1,n})$
 b) $P(L_{n_2,n,n_2}) = (\lambda - 2)^{4n_2} P(L_{n_2,n})$
 c) $P(L_{n_3,n,n_3}) = (\lambda^2 - 5\lambda + 10\lambda - 7)^{n_3} P(L_{n_3,n})$
 a) $P(L_{n_4,n,n_4}) = (\lambda - 2)^{n_4}(\lambda - 3)^{n_4} P(L_{n_4,n})$

The more general ladder-type graphs appear when L_n is sandwiched k times in L_{n_i} , $1 \leq i \leq 4$. We denote these graphs by

$L_{n_1,n,n_1 \dots n_1,n,n_1}$, $L_{n_2,n,n_2 \dots n_2,n,n_2}$, $L_{n_3,n,n_3 \dots n_3,n,n_3}$, and $L_{n_4,n,n_4 \dots n_4,n,n_4}$, and present their chromatic polynomials in the theorem:

Theorem 1.10.

- b) $P(L_{n_1,n,n_1 \dots n_1,n,n_1}) = \lambda(\lambda - 1)(\lambda - 2)^{2(k+1)n_1}(\lambda^2 - 3\lambda + 3)^{kn}$
 c) $P(L_{n_2,n,n_2 \dots n_2,n,n_2}) = \lambda(\lambda - 1)(\lambda - 2)^{4(k+1)n_2}(\lambda^2 - 3\lambda + 3)^{kn}$

d) $P(L_{n_3,n,n_3 \dots n_3,n,n_3}) = \lambda(\lambda - 1)(\lambda^2 - 5\lambda + 10\lambda - 7)^{(k+1)n_3}(\lambda^2 - 3\lambda + 3)^{kn}$

e) $P(L_{n_4,n,n_4 \dots n_4,n,n_4}) = \lambda(\lambda - 1)(\lambda - 2)^{(k+1)n_4}(\lambda - 3)^{(k+1)n_4}(\lambda^2 - 3\lambda + 3)^{kn}$

The chromatic equivalence and chromatic uniqueness of these graphs are reflected in the theorem:

Theorem 1.11.

- a) Neither two of L_n, L_{n_1}, L_{n_3} , and L_{n_4} are chromatically equivalent.
 b) L_{n_1} , and L_{n_2} are chromatically equivalent if $n_1 = 2n_2$, but are not chromatically unique.
 c) $L_{n_1,n}, L_{n_3,n}$ and $L_{n_4,n}$ are not chromatically unique.

2. PROOFS

This section contains the proofs of the results we got

Proof of the theorem 1.5(c). We proceed by induction on n_3 . For $n_3 = 1$ we got

$$\begin{aligned}
 P(\text{Diagram 1}) &= P(\text{Diagram 2}) - P(\text{Diagram 3}) = P(\text{Diagram 4}) - P(\text{Diagram 5}) - P(\text{Diagram 6}) + P(\text{Diagram 7}) = (\lambda - 2)P(\text{Diagram 8}) + \\
 P(\text{Diagram 9}) - P(\text{Diagram 10}) &= (\lambda - 2) \left(P(\text{Diagram 11}) - P(\text{Diagram 12}) \right) + P(\text{Diagram 13}) - P(\text{Diagram 14}) = (\lambda - 2) \left(P(\text{Diagram 15}) - \right. \\
 P(\text{Diagram 16}) - P(\text{Diagram 17}) &+ P(\text{Diagram 18}) + (\lambda - 1)P(\text{Diagram 19}) = (\lambda - 2)2 \left((\lambda - 2)P(\text{Diagram 20}) + P(\text{Diagram 21}) - P(\text{Diagram 22}) \right) + (\lambda - \\
 1)(P(\text{Diagram 23}) - P(\text{Diagram 24})) &= (\lambda - 2) \left((\lambda - 2) \left\{ P(\text{Diagram 25}) - P(\text{Diagram 26}) \right\} + P(\text{Diagram 27}) - P(\text{Diagram 28}) \right) + (\lambda - 1)(P(\text{Diagram 29}) - P(\text{Diagram 30})) = (\lambda - \\
 2) \left((\lambda - 2)(\lambda - 1)P(\text{Diagram 31}) + (\lambda^2 - \lambda) \right) &+ (\lambda - 1)(\lambda^2 - \lambda) = (\lambda - 2) \left((\lambda - 2)(\lambda - 1) \left\{ P(\text{Diagram 32}) - P(\text{Diagram 33}) \right\} + (\lambda^2 - \lambda) \right) + \\
 \lambda(\lambda - 1)^2 &= (\lambda - 2)((\lambda - 2)(\lambda - 1)(\lambda^2 - \lambda) + (\lambda^2 - \lambda)) + \lambda(\lambda - 1)^2 = \lambda(\lambda - 1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7)
 \end{aligned}$$

Now with the assumption that the result holds for an arbitrary n_3 , we have

$$\begin{aligned}
 P(L_{n_3+1}) &= P(\text{Diagram 34}) = P(\text{Diagram 35}) - P(\text{Diagram 36}) \\
 &= P(\text{Diagram 37}) - P(\text{Diagram 38}) - P(\text{Diagram 39}) + P(\text{Diagram 40}) \\
 &= (\lambda - 2)P(\text{Diagram 41}) + P(-P(\text{Diagram 42})) = (\lambda - 2)P(\text{Diagram 43}) - \\
 P(\text{Diagram 44}) + P(\text{Diagram 45}) - P(\text{Diagram 46}) &= (\lambda - 2) \left(P(\text{Diagram 47}) - \right. \\
 P(\text{Diagram 48}) - P(\text{Diagram 49}) &+ P(\text{Diagram 50}) \left. \right) + (\lambda - 1)P(\text{Diagram 51}) = \\
 (\lambda - 2) \left((\lambda - 2)P(\text{Diagram 52}) + P(\text{Diagram 53}) \right) &+ \\
 (\lambda - 1) \left(P(\text{Diagram 54}) - P(\text{Diagram 55}) \right) &= \\
 (\lambda - 2)((\lambda - 2) \left\{ P(\text{Diagram 56}) - P(\text{Diagram 57}) \right\} &+ P(\text{Diagram 58}) - P(\text{Diagram 59})) + \\
 (\lambda - 1)P(\text{Diagram 60}) &= (\lambda - 2) \left((\lambda - 2)(\lambda - 1)P(\text{Diagram 61}) + P(\text{Diagram 62}) \right) = \\
 +(\lambda - 1)P(\text{Diagram 63}) &= ((\lambda - 2)(\lambda^2 - 3\lambda + 3) + (\lambda - 1))P(\text{Diagram 64}) = (\lambda^3 - 3\lambda^2 + 3\lambda - 2\lambda^2 + 6\lambda - 6 + \\
 \lambda - 1)P(L_{n_3}) &= (\lambda^3 - 5\lambda^2 + 10\lambda - 7)(\lambda(\lambda - 1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7))^{n_3+1}, \\
 \text{as was required.} &
 \end{aligned}$$

Proofs of the parts (a), (b), and (d) are similar. □

Proof of Proposition 1.6. Here we give only the proof of part (d); other parts can be proved similarly.

We again prove it by induction on n_8 . For $n_8 = 1$ we have

$$\begin{aligned}
 P(\text{Diagram 65}) &= P(\text{Diagram 66}) - P(\text{Diagram 67}) = P(\text{Diagram 68}) - P(\text{Diagram 69}) - P(\text{Diagram 70}) + P(\text{Diagram 71}) \\
 &= (\lambda - 2)P(\text{Diagram 72}) + P(\text{Diagram 73}) - P(\text{Diagram 74}) = (\lambda - 2) \left\{ P(\text{Diagram 75}) - P(\text{Diagram 76}) \right\} + P(\text{Diagram 77}) \\
 &= (\lambda - 2)(\lambda - 1)P(\text{Diagram 78}) + P(\text{Diagram 79}) = (\lambda^2 - 3\lambda + 3)P(\text{Diagram 80}) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda^2 - 3\lambda + 3)
 \end{aligned}$$

Now with the assumption that the result holds for an arbitrary n_8 , we have

$$2)(\lambda(\lambda-1)(\lambda-2)^{n_8}(\lambda-3)^{n_8}(\lambda^2-3\lambda+3)^{n_8}) = \lambda(\lambda-1)(\lambda-2)^{n_8+1}(\lambda-3)^{n_8+1}(\lambda^2-3\lambda+3)^{n_8+1}. \quad \square$$

For $n = 1$, We get

Suppose the result holds for $n = k$, that is

Now for $n = k + 1$, we have

$$\begin{aligned}
 P\left(\begin{array}{c} \text{Diagram 1} \\ G_1 \end{array}\right) &= P\left(\begin{array}{c} \text{Diagram 2} \\ G_1 \end{array}\right) - P\left(\begin{array}{c} \text{Diagram 3} \\ G_1 \end{array}\right) \\
 &= P\left(\begin{array}{c} \text{Diagram 4} \\ G_1 \end{array}\right) - P\left(\begin{array}{c} \text{Diagram 5} \\ G_1 \end{array}\right) - P\left(\begin{array}{c} \text{Diagram 6} \\ G_1 \end{array}\right) \\
 &\quad + P\left(\begin{array}{c} \text{Diagram 7} \\ G_1 \end{array}\right) \\
 &= (\lambda - 2)P\left(\begin{array}{c} \text{Diagram 8} \\ G_1 \end{array}\right) + P\left(\begin{array}{c} \text{Diagram 9} \\ G_1 \end{array}\right) - P\left(\begin{array}{c} \text{Diagram 10} \\ G_1 \end{array}\right) \\
 &= (\lambda - 2)\left[P\left(\begin{array}{c} \text{Diagram 11} \\ G_1 \end{array}\right) - P\left(\begin{array}{c} \text{Diagram 12} \\ G_1 \end{array}\right) + P\left(\begin{array}{c} \text{Diagram 13} \\ G_1 \end{array}\right)\right] \\
 &= [(\lambda - 2)(\lambda - 1) + 1]P\left(\begin{array}{c} \text{Diagram 14} \\ G_1 \end{array}\right) = [(\lambda - 2)(\lambda - 1) + 1](\lambda^2 - 3\lambda + 1)^k P(G_1) = (\lambda^2 - 3\lambda + 1)^{k+1} P(G_1)
 \end{aligned}$$

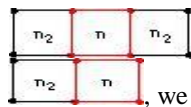
as was required. \square


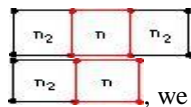
Proof of lemma 1.9. We give only proof of part (b), which is the most difficult; other parts have similar proofs.

Instead of the long ladder



We shall use a short form as



When a single unit () of L_{n_2} is appended to , we get

$$\begin{aligned}
P(L_{n_2, n, 1}) &= P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) = P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) \\
&= P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) \\
&\quad - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) = (\lambda - 2)P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) \\
&= (\lambda - 2)\left[P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right)\right] \\
&= (\lambda - 2)\left[P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right)\right. \\
&\quad \left. - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right)\right] = (\lambda - 2)^2\left[P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right)\right] \\
&= (\lambda - 2)^2\left[P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right)\right. \\
&\quad \left. - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right)\right] = (\lambda - 2)^3P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) \\
&= (\lambda - 2)^3\left[P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right)\right] \\
&= (\lambda - 2)^3\left[P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) - P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right) + P\left(\begin{array}{|c|c|c|c|} \hline n_2 & n & & \\ \hline \end{array}\right)\right] \\
&= (\lambda - 2)^4P\left(\begin{array}{|c|c|} \hline n_2 & n \\ \hline \end{array}\right)
\end{aligned}$$

Now suppose the result holds when the appended ladder L_{n_2} has k units, that is

$$P(L_{n_2, n, k}) = P\left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array}\right) = (\lambda - 2)^{4k}P\left(\begin{array}{|c|c|} \hline n_2 & n \\ \hline \end{array}\right).$$

If the appended ladder has $k + 1$ units, then we receive

$$\begin{aligned}
 P(L_{n_2, n, k+1}) &= P \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) \\
 &= P \left(\begin{array}{c} \text{Diagram 2} \end{array} \right) \\
 &\quad - P \left(\begin{array}{c} \text{Diagram 3} \end{array} \right) \\
 &= P \left(\begin{array}{c} \text{Diagram 4} \end{array} \right) \\
 &\quad - P \left(\begin{array}{c} \text{Diagram 5} \end{array} \right) \\
 &\quad - P \left(\begin{array}{c} \text{Diagram 6} \end{array} \right) \\
 &\quad + P \left(\begin{array}{c} \text{Diagram 7} \end{array} \right) \\
 &= (\lambda - 2) \left[P \left(\begin{array}{c} \text{Diagram 8} \end{array} \right) \right. \\
 &\quad \left. - P \left(\begin{array}{c} \text{Diagram 9} \end{array} \right) \right] \\
 &= (\lambda - 2) \left[P \left(\begin{array}{c} \text{Diagram 10} \end{array} \right) \right. \\
 &\quad - P \left(\begin{array}{c} \text{Diagram 11} \end{array} \right) \\
 &\quad - P \left(\begin{array}{c} \text{Diagram 12} \end{array} \right) \\
 &\quad \left. + P \left(\begin{array}{c} \text{Diagram 13} \end{array} \right) \right] \\
 &= (\lambda - 2)^2 P \left(\begin{array}{c} \text{Diagram 14} \end{array} \right) \\
 &= (\lambda - 2)^2 \left[P \left(\begin{array}{c} \text{Diagram 15} \end{array} \right) \right. \\
 &\quad \left. - P \left(\begin{array}{c} \text{Diagram 16} \end{array} \right) \right] \\
 &= (\lambda - 2)^2 \left[P \left(\begin{array}{c} \text{Diagram 17} \end{array} \right) \right] - P
 \end{aligned}$$

$$\begin{aligned}
& \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) - P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) \\
& + P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) \\
& = (\lambda - 2)^3 P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) \\
& = (\lambda - 2)^3 P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) = (\lambda - 2)^3 \left[P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) \right. \\
& \left. - P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) \right] \\
& = (\lambda - 2)^3 \left[P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) \right. \\
& \left. - P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) - P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) \right. \\
& \left. + P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) \right] \\
& = (\lambda - 2)^4 P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) = (\lambda - 2)^4 (\lambda - 2)^{4k} P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right) \\
& = (\lambda - 2)^{4(k+1)} P \left(\begin{array}{|c|c|c|c|c|c|c|} \hline n_2 & n & & & & & \\ \hline \end{array} \right)
\end{aligned}$$

which is required result.

Proof of the Theorem 1.10. In each case, apply recursively Lemma 1.9 and Theorem 1.5 k times and then use $P(L_{n_1})$.

Proof of the Theorem 1.11. 1. Obvious; just see Theorem 1.5.

2. Simply observe that if $n_1 = 2n_2$, then $P(L_{n_1}) = P(L_{n_2})$. These are not chromatically unique because $L_{n_1} \neq L_{n_2}$; just observe that there are vertices of degree 5 in L_{n_2} but are not in L_{n_1} .

3. It is obvious; simply observe that $P(L_{n_1,n}) = P(L_{n_5})$, $P(L_{n_2,n}) = P(L_{n_6})$, $P(L_{n_3,n}) = P(L_{n_7})$, and $P(L_{n_4,n}) = P(L_{n_8})$ while $L_{n_1,n} \not\cong L_{n_5}$, $L_{n_2,n} \not\cong L_{n_6}$, $L_{n_3,n} \not\cong L_{n_7}$, $L_{n_4,n} \not\cong L_{n_8}$.

REFERENCES

- [1] Birkhoff, G. D., "A Determinant Formula for the Number of Ways of Coloring a Map," *Annals of Mathematics*, **14**, 42-46, (1912)
- [2] Bollobás, B., "Modern Graph Theory," *Springer*, New York, (1998)
- [3] Chartrand, G., Lesniak, L., Zhang, P., "Graphs and Digraphs," *CRC Press, Chapman and Hall Book*, Boca Raton, (2011)
- [4] Chia, G. L., "A Bibliography on Chromatic Polynomials," *Discrete Mathematics*, **172**, 175-191, (1997)
- [5] Diestel, R., "Graph Theory," *Springer*, 1997.
- [6] Dong, F. M., Koh, K. M., Teo, K. L., "Chromatic Polynomials and Chromaticity of Graphs," *World Scientific*, New jersey, (2005)
- [7] Whitney, H., "A logical Expression in Mathematics," *Bull. Amer. Math. Soc*, **38**, 572-579, (1932)