ON CHROMATICITY OF LADDER-TYPE GRAPHS

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ABSTRACT: *We give general formulas of chromatic polynomial of some interesting families of Ladder-Type graphs, and conclude that, except two, neither two of them are chromatically equivalent. Moreover, some of them are not chromatically unique.*

Mathematics Subject Classification: *Primary 05C31; Secondary 57M27*
Keywords: *Chromatic polynomial, Chromatic equivalence, Chromatic uniqueness, λ-coloring, Ladder-type graph*

1. INTRODUCTION

The chromatic polynomial was introduced by G. D. Birkhoff in 1912 as a function that counts the number of graph colorings for planer graphs to solve the four color problem [1]. In 1932 H. Whitney generalized it from the planer graphs to the arbitrary graphs [7]. The chromatic polynomial, because of its theoretical and applied importance, has generated a large body of work. Chia [4] provides an extensive bibliography on the chromatic polynomial, and Dong, Koh, and Teo [6] gave a comprehensive treatment.

The following two operations are essential to understand the chromatic polynomial definition for a graph G. These are *edge deletion*, denoted by $G'' = G - e$, and *edge contraction*, *denoted by* $G'' = G/e$.

Definition 1.1. The chromatic polynomial is a function P from the set of graphs to the set $\mathbb{Z}[\lambda]$, a ring of polynomials, such that

$$P(G) = \begin{cases} 0 & \text{if there is a loop in } G \\ \lambda^n & \text{if } G \text{ consists of only } n \text{ isolated vertices} \\ P(G-e) - P(G/e) & \text{otherwise} \end{cases}$$

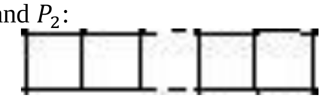
Two graphs are chromatically equivalent if they have the same chromatic polynomial; a graph G is chromatically unique if $P(G) = P(G')$ implies $G \cong G'$.

For a positive integer λ, a λ-coloring of a graph G is a mapping of $V(G)$ into the set $\{1,2,3,\dots,\lambda\}$ of λ colors. Thus, there are exactly λ^n colorings for a graph on n vertices. If φ is a λ-coloring such that $\varphi(u) \neq \varphi(v)$ for all $uv \in E$, then φ is called a proper (or admissible) coloring. The chromatic number of a graph G, denoted by $\gamma(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

Remark 1.2. Every evaluation of chromatic polynomial at some number λ actually gives the λ-coloring of the graph.

Since we are interested mainly in ladder-type graphs, we define them here. First, the two closely related definitions:

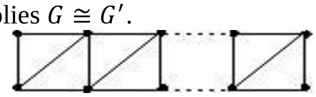
Definition 1.3. A ladder graph L_n is the Cartesian product of path graphs P_n and P_2:



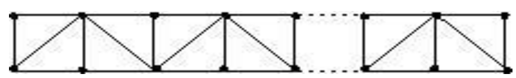
$L_n = P_n \times P_2$

We define a *ladder-type graph* a ladder graph with addition of some edges and vertices, in some pattern, keeping the main structure of L_n intact.

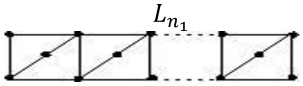
The ladder type graphs we are concerned with are:



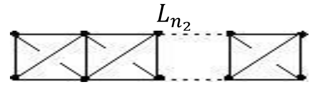
L_{n_1}



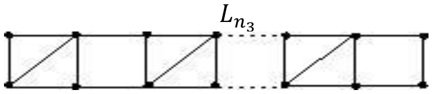
L_{n_2}



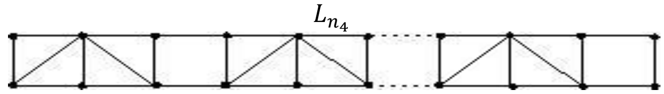
L_{n_3}



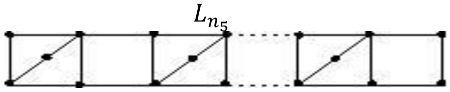
L_{n_4}



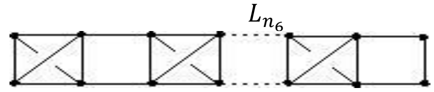
L_{n_5}



L_{n_6}

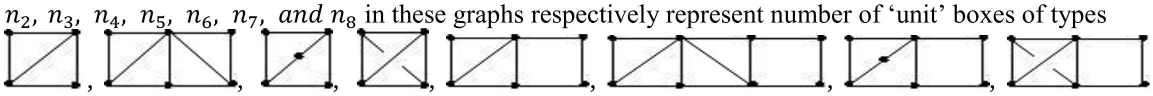


L_{n_7}



L_{n_8}

The subscript $n_1,\ n_2,\ n_3,\ n_4,\ n_5,\ n_6,\ n_7,\ and\ n_8$ in these graphs respectively represent number of 'unit' boxes of types



The following is the chromatic polynomial of the ladder graph L_n, which already exists in the literature.

Proposition 1.4. The chromatic polynomial of the graph L_n, is

$$P(L_n) = \lambda(\lambda-1)(\lambda^2 - 3\lambda + 3)^n.$$

First we give the chromatic polynomials of four 'basic' ladder-type graphs:

Theorem 1.5. The chromatic polynomials of L_{n1}, L_{n2}, $L_{n3,}$ *and* L_{n4} are

a) $P(L_{n_1}) = \lambda(\lambda-1)(\lambda-2)^{2n_1}$,
b) $P(L_{n_2}) = \lambda(\lambda-1)(\lambda-2)^{4n_2}$,
c) $P(L_{n_3}) = \lambda(\lambda-1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7)^{n_3}$,
d) $P(L_{n_4}) = \lambda(\lambda-1)(\lambda-2)^{n_4}(\lambda-3)^{n_4}$

Then we have the proposition:

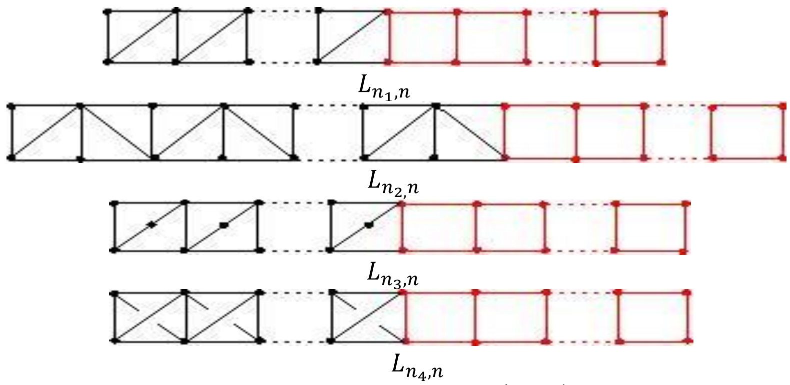
Proposition 1.6. The chromatic polynomials of $L_{n_5}, L_{n_6}, L_{n_7},\ and\ L_{n_8}$ are

a) $P(L_{n_5}) = \lambda(\lambda-1)(\lambda-2)^{2n_5}(\lambda^2 - 3\lambda + 3)^{n_5}$
b) $P(L_{n_6}) = \lambda(\lambda-1)(\lambda-2)^{2n_6}(\lambda^2 - 3\lambda + 3)^{n_6}$

c) $P(L_{n_7}) = \lambda(\lambda-1)(\lambda^2-3\lambda+3)^{n_7}(\lambda^3-5\lambda^2+10\lambda-7)^{n_7}$, and

d) $P(L_{n_6}) = \lambda(\lambda-1)(\lambda-2)^{n_8}(\lambda-3)^{n_8}(\lambda^2-3\lambda+3)^{n_8}$

Besides the above graphs, the following are special types of ladder-types graphs. These are actually obtained by appending the ladder graph L_n to the graphs $L_{n_1}, L_{n_2}, L_{n_3}, and\ L_{n_4}$.



We shall give the chromatic polynomials of these graphs as a corollary of the general result:

Theorem 1.7. If a graph G is obtained by appending L_n to a graph G_1 such that they share nothing except just one edge, then

$$P(G) = (\lambda^2-3\lambda+3)^n P(G_1).$$

Corollary 1.8.

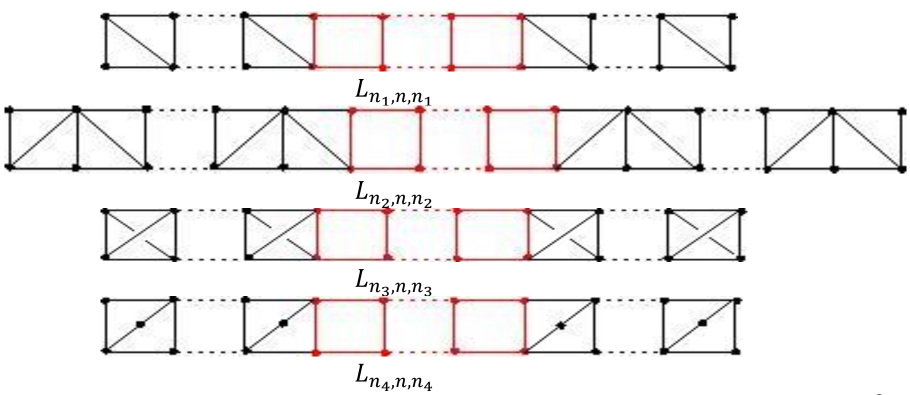
a) $P(L_{n_1,n}) = \lambda(\lambda-1)(\lambda-2)^{2n_1}$

b) $P(L_{n_2,n}) = \lambda(\lambda-1)(\lambda-2)^{4n_2}[\lambda^2-3\lambda+3]^n$.

c) $P(L_{n_3,n}) = \lambda(\lambda-1)(\lambda^3-5\lambda^2+10\lambda-7)^{n_3}[\lambda^2-3\lambda+3]^n$

d) $P(L_{n_4,n}) = \lambda(\lambda-1)(\lambda-2)^{n_4}(\lambda-3)^{n_4}[\lambda^2-3\lambda+3]^n$

If L_n is sandwiched between ladder-type graph L_{n_i}, $1 \le i \le 4$, then we shall denote the resultant ladder-type graph by L_{n_i,n,n_i}. The chromatic polynomial of the following graphs are given in a lemma:



Lemma 1.9.

a) $P(L_{n_1,n,n_1}) = (\lambda-2)^{2n_1}P(L_{n_1,n})$

b) $P(L_{n_2,n,n_2}) = (\lambda-2)^{4n_2}\,P(L_{n_2,n})$

c) $P(L_{n_3,n,n_3}) = (\lambda^2-5\lambda+10\lambda-7)^{n_3}P(L_{n_3,n})$

a) $P(L_{n_4,n,n_4}) = (\lambda-2)^{n_4}(\lambda-3)^{n_4}P(L_{n_4,n})$

The more general ladder-type graphs appear when L_n is sandwiched k times in L_{n_i}, $1 \le i \le 4$. We denote these graphs by $L_{n_1,n,n_1\ldots n_1,n,n_1}$, $L_{n_2,n,n_2\ldots n_2,n,n_2}$, $L_{n_3,n,n_3\ldots n_3,n,n_3}$, $and\ L_{n_4,n,n_4\ldots n_4,n,n_4}$, and present their chromatic polynomials in the theorem:

Theorem 1.10.

b) $P(L_{n_1,n,n_1\ldots n_1,n,n_1}) = \lambda(\lambda-1)(\lambda-2)^{2(k+1)n_1}(\lambda^2-3\lambda+3)^{kn}$

c) $P(L_{n_2,n,n_2\ldots n_2,n,n_2}) = \lambda(\lambda-1)(\lambda-2)^{4(k+1)n_2}(\lambda^2-3\lambda+3)^{kn}$

d) $P(L_{n_3,n,n_3\ldots n_3,n,n_3}) = \lambda(\lambda-1)(\lambda^2-5\lambda+10\lambda-7)^{(k+1)n_3}(\lambda^2-3\lambda+3)^{kn}$

e) $P(L_{n_4,n,n_4\ldots n_4,n,n_4}) = \lambda(\lambda-1)(\lambda-2)^{(k+1)n_4}(\lambda-3)^{(k+1)n_4}(\lambda^2-3\lambda+3)^{kn}$

The chromatic equivalence and chromatic uniqueness of these graphs are reflected in the theorem:

Theorem 1.11.

a) Neither two of $L_n, L_{n_1}, L_{n_3}, and\ L_{n_4}$ are chromatically equivalent.

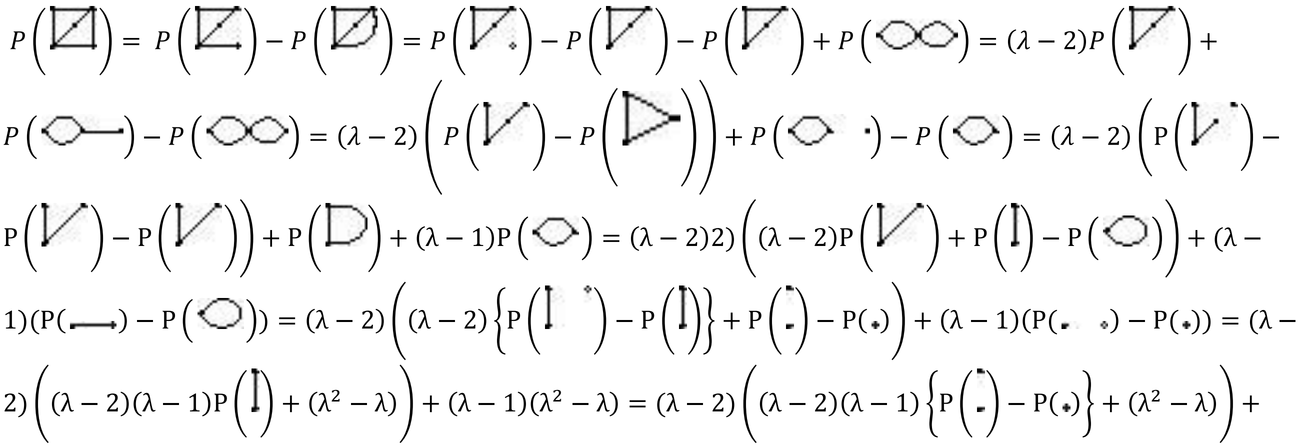
b) $L_{n_1}, and\ L_{n_2}$ are chromatically equivalent if $n_1 = 2n_2$, but are not chromatically unique.

c) $L_{n_1,n}, L_{n_3,n}\ and\ L_{n_4,n}$ are not chromatically unique.

2. PROOFS

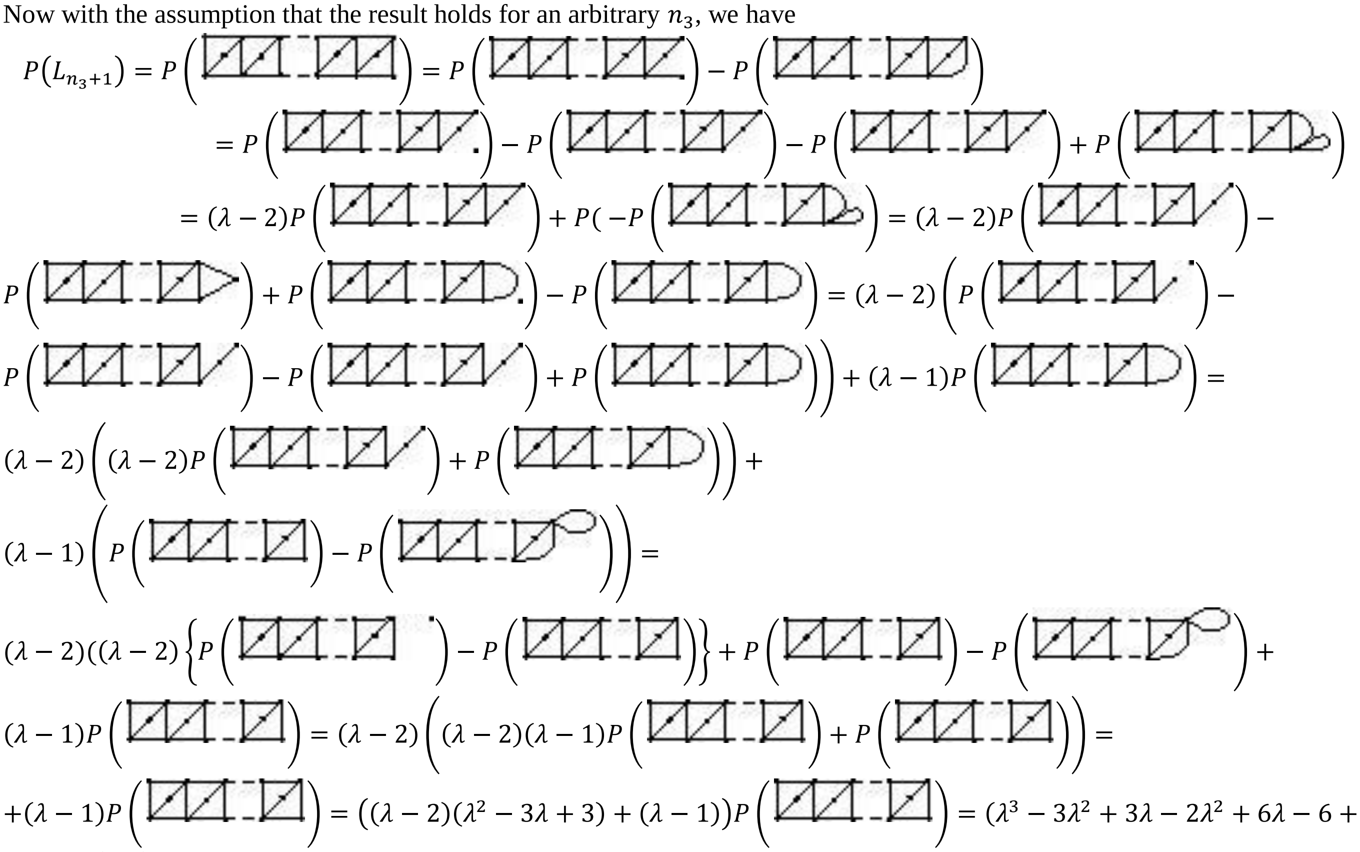
This section contains the proofs of the results we got

Proof of the theorem 1.5(c). We proceed by induction on n_3. For $n_3 = 1$ we got



$\lambda(\lambda-1)^2 = (\lambda-2)\big((\lambda-2)(\lambda-1)(\lambda^2-\lambda)+(\lambda^2-\lambda)\big)+\lambda(\lambda-1)^2 = \lambda(\lambda-1)(\lambda^3-5\lambda^2+10\lambda-7)$

Now with the assumption that the result holds for an arbitrary n_3, we have



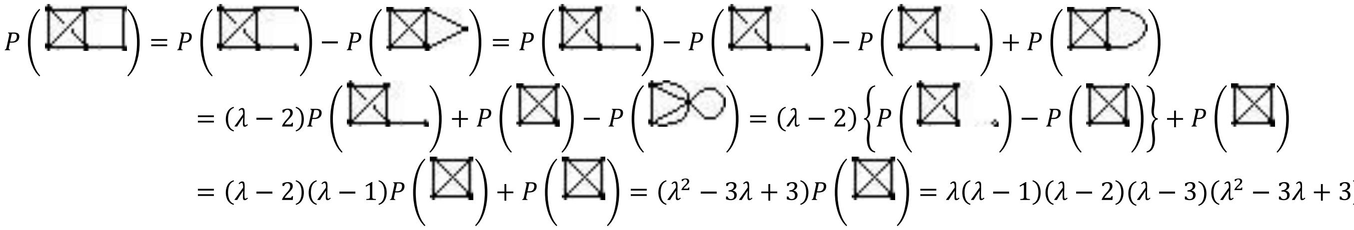
$\lambda-1)P(L_{n_3}) = (\lambda^3-5\lambda^2+10\lambda-7)(\lambda(\lambda-1)(\lambda^3-5\lambda^2+10\lambda-7)^{n_3+1}$,

as was required.

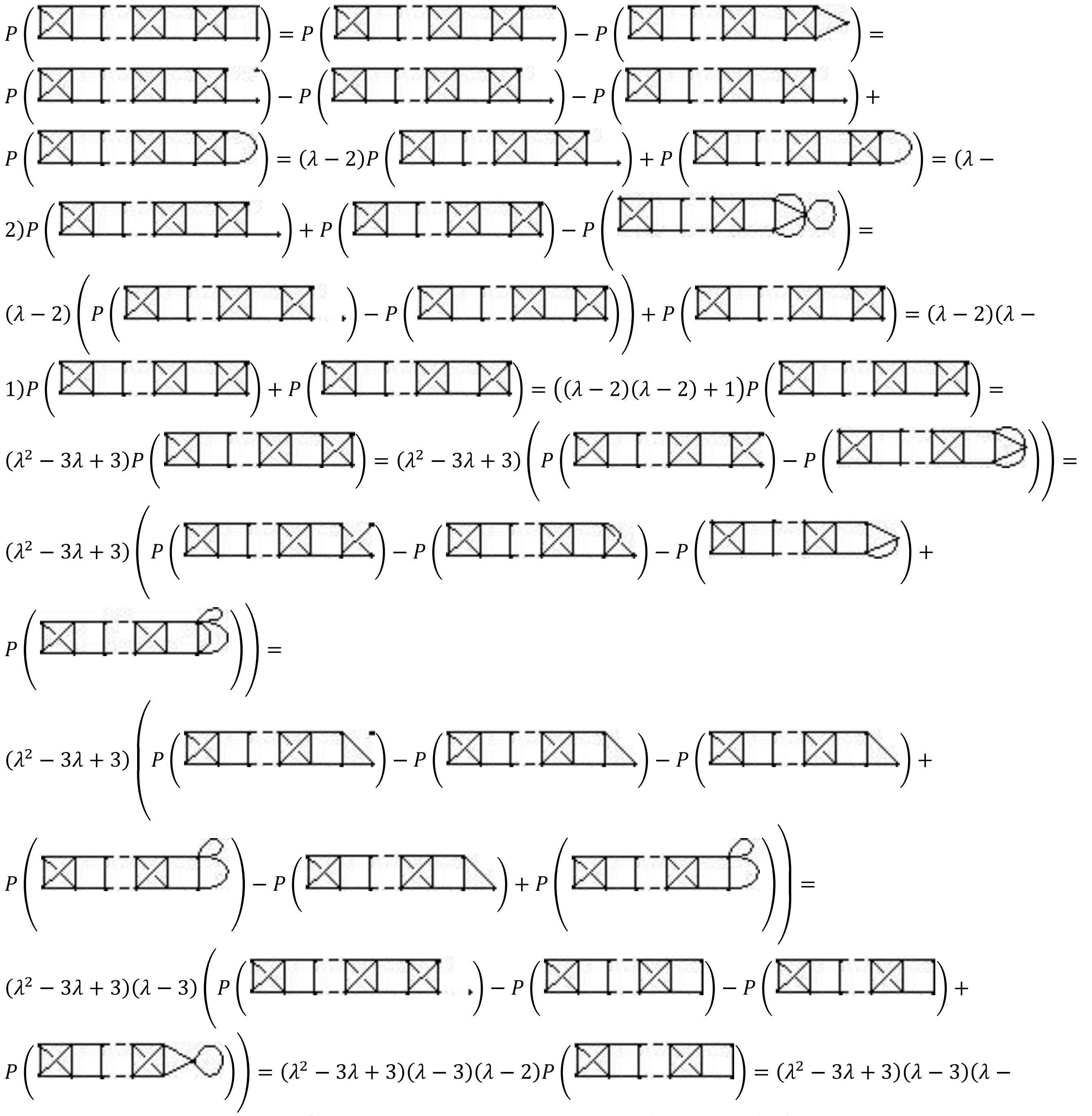
Proofs of the parts (a), (b), and (d) are similar. □

Proof of Proposition 1.6. Here we give only the proof of part (d); other parts can be proved similarly.

We again prove it by induction on n_8. For $n_8 = 1$ we have



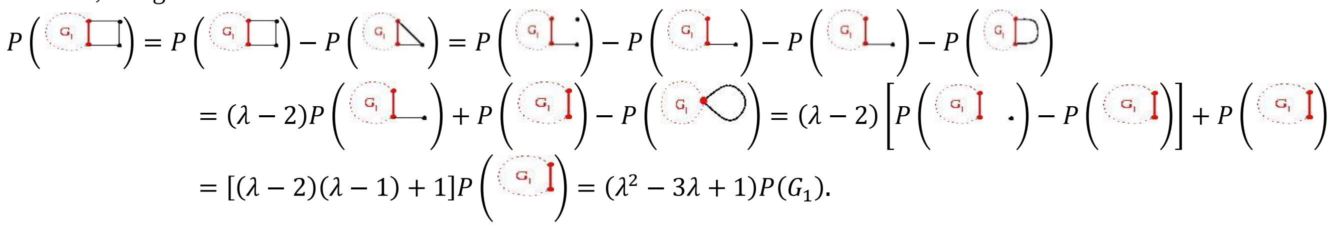
Now with the assumption that the result holds for an arbitrary n_8, we have



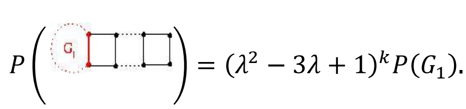
$2)(\lambda(\lambda-1)(\lambda-2)^{n_8}(\lambda-3)^{n_8}(\lambda^2-3\lambda+3)^{n_8}) = \lambda(\lambda-1)(\lambda-2)^{n_8+1}(\lambda-3)^{n_8+1}(\lambda^2-3\lambda+3)^{n_8+1}$. □

Proof of Theorem 1.7. We proceed induction on n:

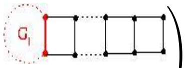
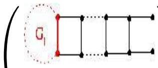
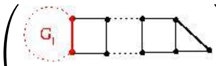
For $n = 1$, We get

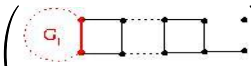
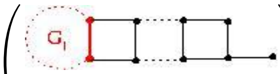
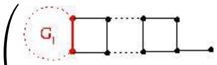


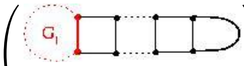
Suppose the result holds for $n = k$, that is

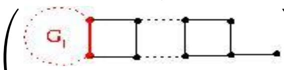
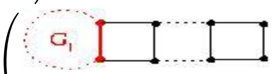
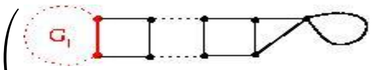


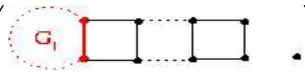
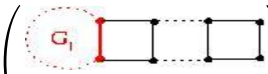
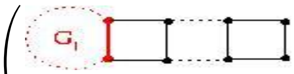
Now for $n = k + 1$, we have

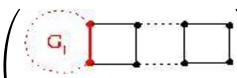
$P\Big($  $\Big) = P\Big($  $\Big) - P\Big($  $\Big)$

$= P\Big($  $\Big) - P\Big($  $\Big) - P\Big($  $\Big)$

$+ P\Big($  $\Big)$

$= (\lambda - 2)P\Big($  $\Big) + P\Big($  $\Big) - P\Big($  $\Big)$

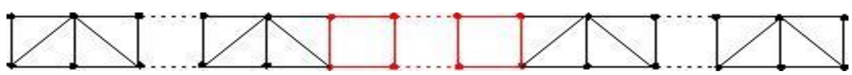
$= (\lambda - 2)[P\Big($  $\Big) - P\Big($  $\Big) + P\Big($  $\Big)$

$= [(\lambda - 2)(\lambda - 1) + 1]P\Big($  $\Big) = [(\lambda - 2)(\lambda - 1) + 1](\lambda^2 - 3\lambda + 1)^k P(G_1) = (\lambda^2 - 3\lambda + 1)^{k+1} P(G_1)$

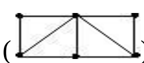
as was required. □

Proof of lemma 1.9. We give only proof of part (b), which is the most difficult; other parts have similar proofs.

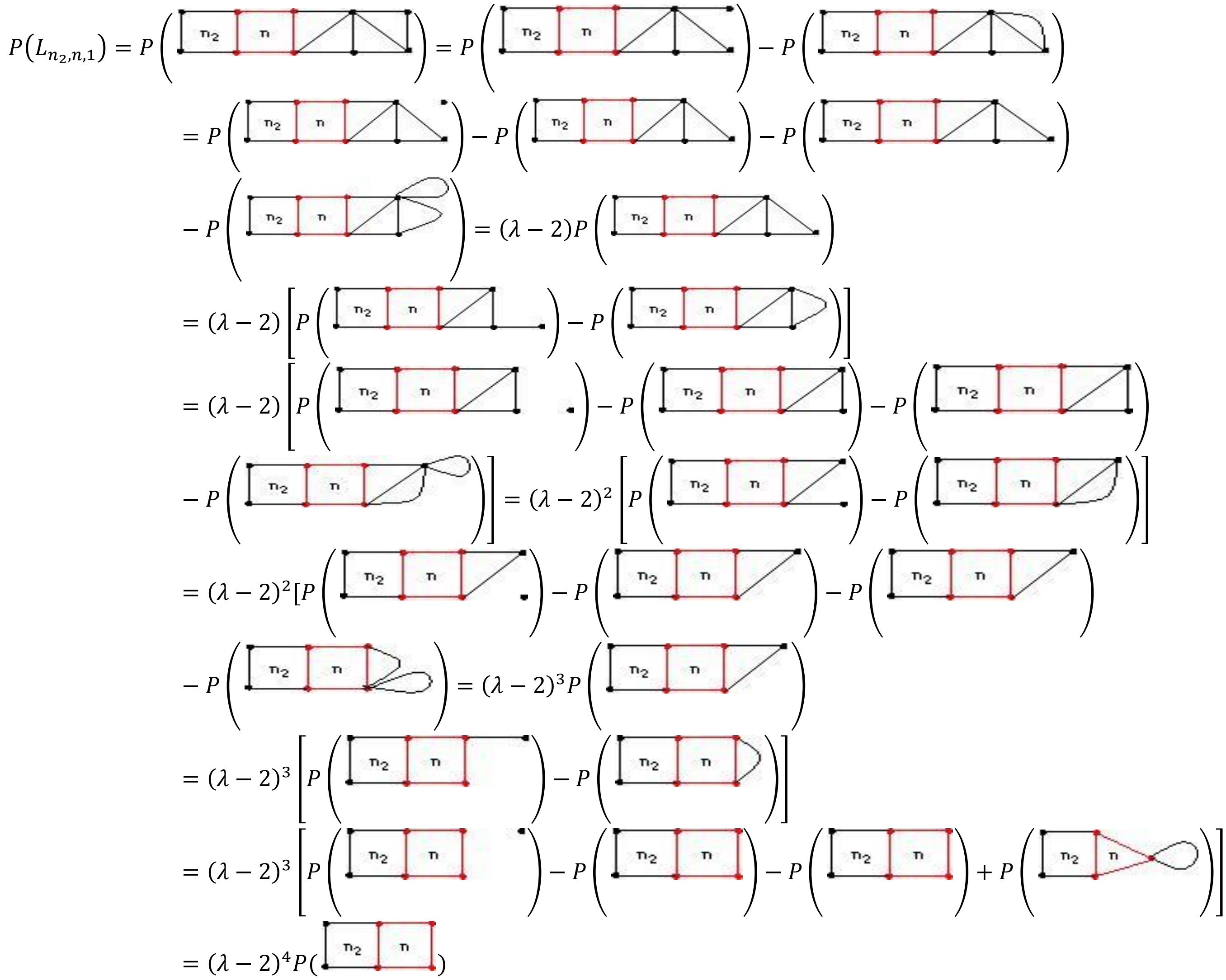
Instead of the long ladder



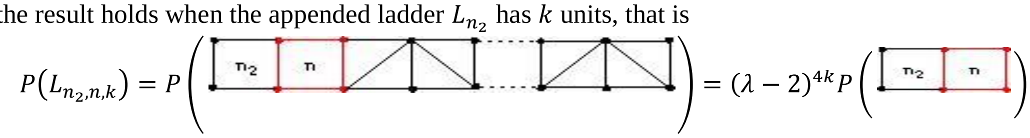
We shall use a short form as

When a single unit () of L_{n_2} is appended to 

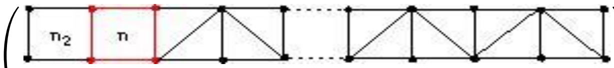
, we get

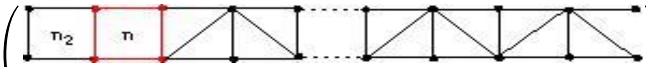


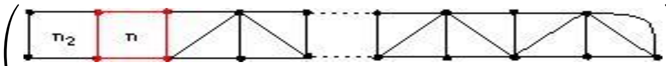
Now suppose the result holds when the appended ladder L_{n_2} has k units, that is

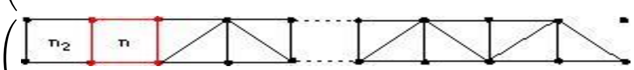


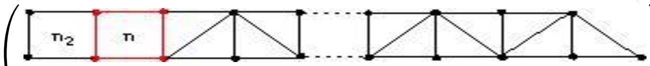
If the appended ladder has $k + 1$ units, then we recieve

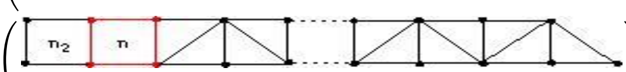
$P(L_{n_2,n,k+1}) = P$ ()

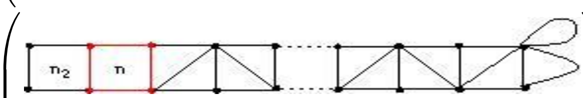
$= P$ ()

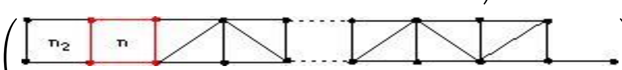
$- P$ ()

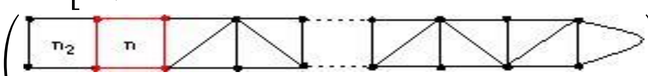
$= P$ ()

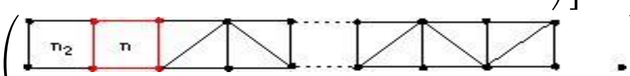
$- P$ ()

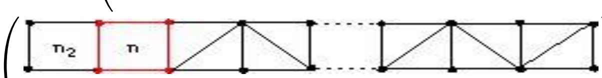
$- P$ ()

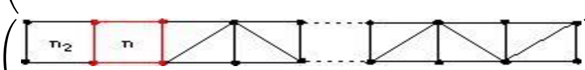
$+ P$ ()

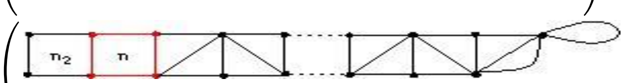
$= (\lambda - 2)$ [P ()

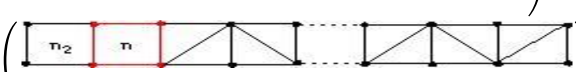
$- P$ ()]

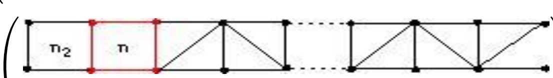
$= (\lambda - 2)$[P ()

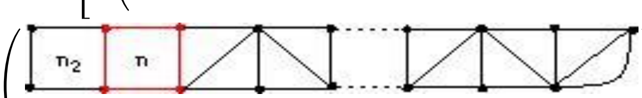
$- P$ ()

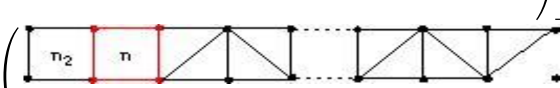
$- P$ ()

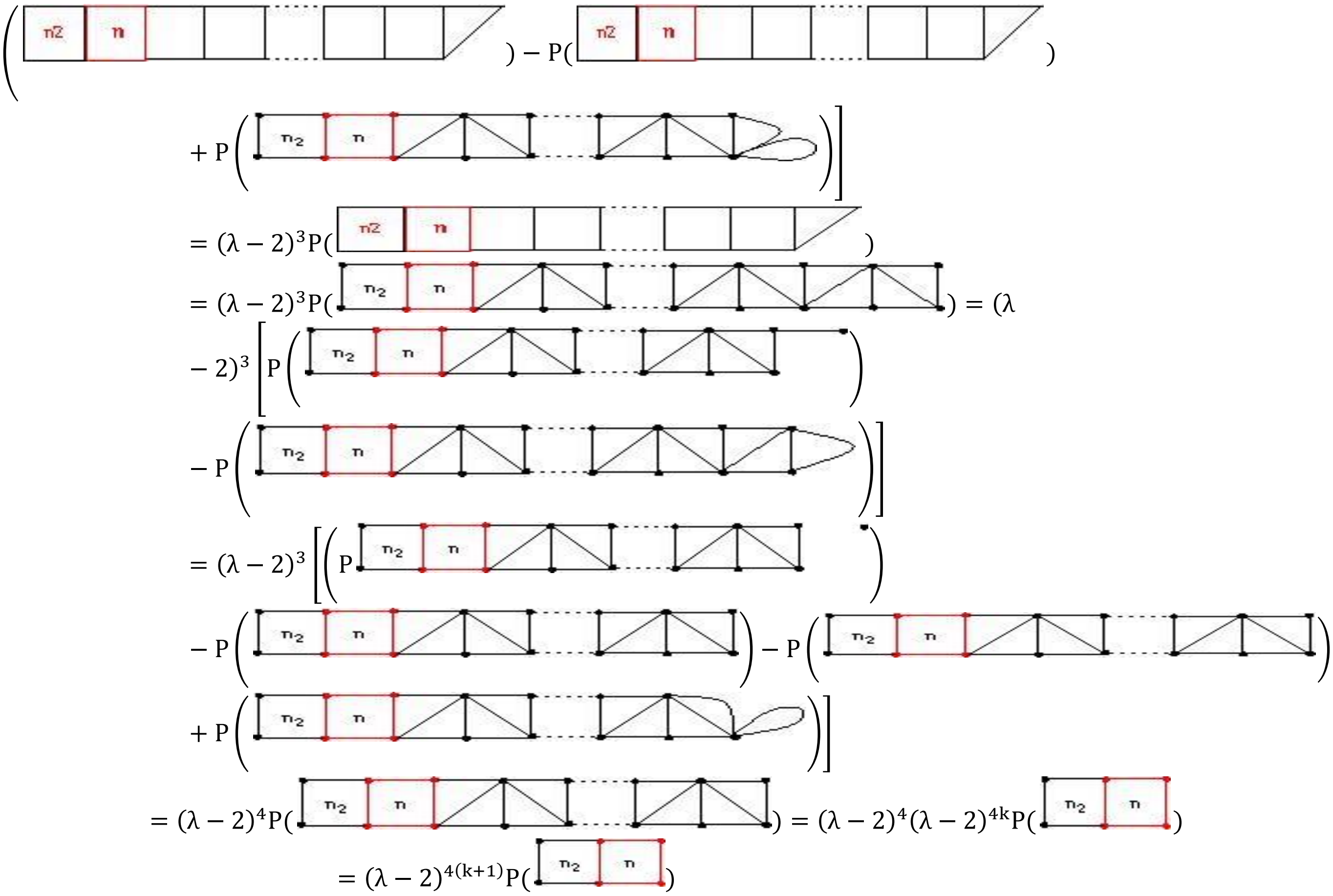
$+ P$ ()

$= (\lambda - 2)^2 P$ ()

$= (\lambda - 2)^2$ [P ()

$- P$ ()]

$= (\lambda - 2)^2$[P () $- P$



which is required result.

Proof of the Theorem 1.10. In each case, apply recursively Lemma 1.9 and Theorem 1.5 k times and then use $P(L_{n_1})$.

Proof of the Theorem 1.11. 1. Obvious; just see Theorem 1.5.

2. Simply observe that if $n_1 = 2n_2$, then $P(L_{n_1}) = P(L_{n_2})$. These are not chromatically unique because $L_{n_1} \neq L_{n_2}$; just observe that there are vertices of degree 5 in L_{n_2} but are not in L_{n_1}.

3. It is obvious; simply observe that $P(L_{n_1,n}) = P(L_{n_5})$, $P(L_{n_2,n}) = P(L_{n_6})$, $P(L_{n_3,n}) = P(L_{n_7})$, and $P(L_{n_4,n}) = P(L_{n_8})$ while $L_{n_1,n} \ncong L_{n_5}, L_{n_2,n} \ncong L_{n_6}, L_{n_3,n} \ncong L_{n_7}, L_{n_4,n} \ncong L_{n_8}$.

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