# SOLVING DIFFERENT ORDER BOUNDARY VALUE PROBLEMS USING OPTIMAL HOMOTOPY ASYMPTOTIC METHOD

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***ABSTRACT:*** *In this paper, the method, Optimal Homotopy Asymptotic Method (OHAM) is applied to solve different order boundary value problems. This method gives a series solution whose convergence is restrained optimally and the convergence region can be adjusted according to the problem concerned. Numerical results are compared with the results obtained by using Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM).The results show that the suggested method is more effective and easy to use.*

**Keywords:** Different order boundary value problems. OHAM, HPM, HAM

## ..INTRODUCTION

Differential equations can be used to model different physical systems such as sociological, economical, biological and chemical etc. Also in literature physical problems are investigated by differential equations, which are mostly handled by the common methods, variational iteration method (VIM), Adomian decomposition method(ADM), Splines (S), Homotopy perturbation method (HPM).The non perturbed techniques (DTM) and ADM) concern nonlinear problems but the region of convergence of their series solution is generally small. Recently Herisanu and Marinca et al.[3-5] introduced OHAM for approximate solution of nonlinear problems of thin film flow of a fourth grade fluid down a vertical cylinder. They used OHAM for understanding the behaviour of nonlinear mechanical vibration of an electrical machine. By using this method they investigated solution of nonlinear equations arising in the study of state flow of a fourth grade fluid past a porous plate. This method supplies the need to control the convergence. In general the OHAM solution agrees with the exact solution. The graphs of the two solutions are coincident. We have applied OHAM to various types of boundary value problems and have investigated that the original exact solution agrees with the numerical solution, the error noted is small .This method is effective and easy to use.

## 2. Analysis of the method

Considering the following differential equation

$$L(F(y))+g(y)+N(F(y))=0\,,B\left(F,\frac{dF}{dy}\right)=0 \quad (2.1a)$$

Where $L$ is taken as linear operator $y$ is independent variable $F(y)$ is an unknown function $g(y)$ is a known function, $N(F(y))$ is a nonlinear operator and $B$ is a boundary operator. According to the idea of OHAM we construct a Homotopy as given below

$$H(\theta(y,p),p):R\times[0,1]\to R$$

That satisfies

$$(1-p)[L(\theta(y,p))+g(y)]=h(p)[L(\theta(y,p))+g(y)+N(\theta(y,p))],B\left(\theta(y,p),\frac{\partial(\theta(y,p)}{\partial y}\right)=0 \quad (2.2a)$$

Where $y\in R$, $p\in[0,1]$ is an embedding parameter, $h(p)$ is a nonzero auxiliary function for $p\neq 0$, $h(0)=0$ and $\theta(y,p)$ is an unknown function. Evidently, for p=0 and p=1 it restrains that unknown function. Evidently, for p=0 and p=1 it restrains that $\theta(y,1)=F(y)$ respectively. Thus as $p$ varies from $0$ to $1$, the solution $\theta(y,p)$ approaches from $F_0(y)$ to $F_1(y)$ where $F_0(y)$ is obtained from Equation (2.2a) for $p=0$ and we have.

$$L(F_0(y))+g(y)=0\,,B\left(F_0,\frac{dF_0}{dy}\right)=0 \quad (2.3a)$$

Now choosing auxiliary function $h(p)$ in the following pattern

$$h(p)=pC_1+p^2C_2+\dots\dots\dots \quad (2.4a)$$

Where $C_1,C_2\cdots$ are constants to be determined later . $h(p)$ can be expressed in many forms as investigated by V. Marinca [3-5] .To get an approximate solution one can expand $\theta(y,p,C_i)$ in Taylor's series about p in the following pattern.

$$\theta(y,p,C_i)=F_0(y)+\sum_{k=1}^{\infty}F_k(y,C_1,C_2,...C_k)p^k \quad (2.5a)$$

Making use of equation (2.5a) in equation (2.2a) and comparing the coefficients of like powers of $p$ we have the following linear equations zeroth order problem is given by equation (2.3a) and the first and second order problems are given by equations (2.6a) and (2.7a)

$$L(F_1(y))+g(y)=C_1N_0(F_0(y)),B\left(F_1,\frac{dF_1}{dy}\right)=0 \quad (2.6a)$$

$$L(F_2(y)) - L(F_1(y)) = C_2 N_0(F_0(y)) + C_1[L(F_1(y)) + N_1(F_0(y), F_1(y))], B\left(F_2, \frac{dF_2}{dy}\right) = 0 \quad (2.7a)$$

The general governing equations for $F_k(y)$ are given by

$$L(F_k(y)) - L(F_{k-1}(y)) = C_k N_0(F_0(y)) + \sum_{i=1}^{k-1} C_i[L(F_{k-i}(y)) + N_{k-i}(F_0(y), F_0(y),. .., F_{k-1}(y)), k = 2,3..., B\left(F_k, \frac{dF_k}{dy}\right) = 0 \quad (2.8a)$$

Where $N_m(F_0(y), F_1(y)..., F_{k-1}(y))$ is the coefficient of $p^m$ in the expansion of $N(\theta(y, p))$ about the embedding Parameter? $p$

$$N(\theta(y, p, C_i)) = N_0(F_0(y)) + \sum_{m=1}^{\infty} N_m(F_0, F_1...F_m) p^m \quad (2.9a)$$

It has been investigated that the convergence of the series (2.5a) depends on the auxiliary constants $C_1$ , $C_2$ ,.. If it is convergent. At $p = 1$ then we have

$$\theta(y, C_i) = F_0(y) + \sum_{k=1}^{\infty} F_k(y, C_1, C_2...C_k) \quad (2.10a)$$

The result of the mth order approximations are

$$\tilde{F}(y, C_1, C_2, ...C_m) = F_0(y) + \sum_{i=1}^{m} F_i(y, C_1, C_2, ...C_k) \quad (2.11a)$$

Using equation (2.11a) into equation (2.1a) we get the residual

$$R(y, C_1, C_2..., C_m) = L(\tilde{F}(y, C_1, C_2..., C_m)) + g(y) + N(\tilde{F}(y, C_1, C_2..., C_m)) \quad (2.12a)$$

If $R = 0$ then $\tilde{F}$ be the exact solution. Generally it does not happen especially in nonlinear problems .In order to get the Optimal Values of $C_i$'s i=1,2,3… we first construct the Functional Values of $C_i$'s i=1,2,3… we first construct the Functional

$$J(C_1, C_2, ..., C_m) = \int_a^b R^2(y, C_1, C_2, ..., C_m) dy \quad (2.13a)$$

And then minimizing it we get

$$\frac{\partial J}{\partial C_1} = 0, \frac{\partial J}{\partial C_2} = 0, ..., \frac{\partial J}{\partial C_m} = 0 \quad (2.14a)$$

Knowing the values of $C_1, C_2 \cdots C_m$. The approximate solution of order m is determined. Where a, b lie in domain of the concerned problem using the least square method we get OHAM solution.

## 3. Numerical problems

**Problem3.1.** [First order linear]

Consider the following linear differential equation

$$F'(y) - F(y) - y\cos y + y\sin y - \sin y = 0,$$
$$F(0) = 0 \quad (3.1b)$$

The exact solution of the problem is

$$F(y) = y\sin y \quad (3.2b)$$

Applying the method mentioned in section 2, the zeroth order problem is

$$F_0'(y) = 0, F_0(0) = 0 \quad (3.3b)$$

Its solution is

$$F_0(y) = 0 \quad (3.4b)$$

First order problem is

$$F_1'(y, C_1) = -y\cos y C_1 - \sin y C_1 + y\sin y C_1 - C_1 F_0(y) + (1 + C_1) F_0'(y), F_1(0) = 0 \quad (3.5b)$$

Its solution is

$$F_1(y, C_1) = -y\cos y C_1 + \sin y C_1 - y\sin y C_1 \quad (3.6b)$$

(3.6b)

Second order problem is

$$F_2'(y, C_1, C_2) = \begin{pmatrix} -y\cos y C_2 - \sin y C_2 + y\sin y C_2 - \\ C_2 F_0(y) - C_1 F_1(y) + C_2 F_0'(y) + (1 + C_1) F_1' \\ F_2(0) = 0 \end{pmatrix}$$

(3.7b)

Its solution is

$$F_2(y, C_1, C_2) = -y\cos y C_1 + \sin y C_1 - y\sin y C_1 - 2C_1^2 + 2\cos y C_1^2 - 2y\cos y C_1^2 + 2\sin y C_1^2 - y\cos y C_2 + \sin y C_2 - y\sin y C_2$$

(2.11a)

(3.8b)

Third order problem is

$$F_3'(y, C_1, C_2, C_3) = -y\cos y C_3 - \sin y C_3 + y\sin y C_3 - C_3 F_0(y) - C_2 F_1(y) - C_1 F_2(y) + C_3 F_0'(y) + C_2 F_1'(y) + (1 + C_1) F_2'(y), F_3(0)$$

(3.9b)

Its solution is

$$F_3(y, C_1, C_2, C_3) = -y\cos y C_1 + \sin y C_1 - y\sin y C_1 - 4C_1^2 + 4\cos y C_1^2 - 4y\cos y C_1^2 + 4\sin y C_1^2 - 6C_1^3 + 2y C_1^3 + 6\cos y C_1^3 - 2y\cos y C_1^3 + 2y\sin y C_1^3 - y\cos y C_2 + \sin y C_2 - y\sin y C_2 - 4C_1C_2 + 4\cos y C_1 C_2 - 4y\cos y C_1 C_2 + 4\sin y C_1 C_2 - y\cos y C_3 + \sin y C_3 - y\sin y C_3 \quad (3.10b)$$

Now we use equations (3.4b), (3.6b), (3.8b), (3.10b), the third order approximate solution by OHAM for p=1 is

$$\tilde{F}(y,C_1,C_2,C_3) = F_0(y) + F_1(y,C_1) + F_2(y,C_1,C_2) + F_3(y,C_1,C_2,C_3) \qquad (3.11b)$$

Using the technique mentioned in section 2 on the domain $a=0, b=1.$ we use the residual

$$R = \tilde{F}' - \tilde{F} - y\cos y + y\sin y - \sin y \qquad (3.12b)$$

The following values of $C_1, C_2, C_3$ are obtained

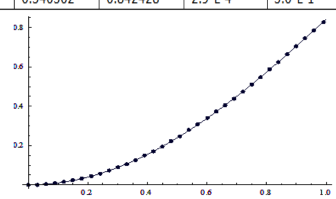
$$C_1 = -1.1763684395430054, C_2 = 0.11688800369849017, C_3 = -0.0026970142170896472 \qquad (3.13b)$$

Considering the values of $C_1, C_2, C_3$, the approximate solution becomes

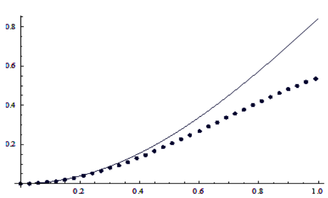
$$\begin{aligned}\tilde{F}(y) = {} & 1.01231\, y^2 - 0.0480322\, y^3 - 0.105409\, y^4 - \\ & 0.0223286\, y^5 + 0.00421533\, y^6 + 0.00112045\, y^7 \\ & -0.0000878039\, y^8 - 0.0000237398\, y^9 + \\ & 1.11482\times10^{-6}\, y^{10} + 2.90161\times10^{-7}\, y^{11} \\ & -9.50035\times10^{-9}\, y^{12} - 2.33658\times10^{-9}\, y^{13} + \\ & 5.79949\times10^{-11}\, y^{14} + 1.3396\times10^{-11}\, y^{15} + O(y^{16})\end{aligned}$$

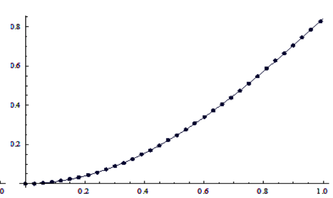
The following table 3.1 displays values of exact solution (3.2b) OHAM solution (3.14b) and the error.

| $y$ | Exact sol | OHAM sol | HPM sol | HAM sol | Er OHAM | Er HPM | Er HAM |
|---|---|---|---|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 | 0.0000000 | 0.0 E-0 | 0.0 E-0 | 0.0 E-0 |
| 0.1 | 0.00998334 | 0.0100643 | 0.00965034 | 0.00999199 | 8.0 E-6 | 3.3 E-4 | 8.0 E-6 |
| 0.2 | 0.0397379 | 0.0399325 | 0.0370779 | 0.0397289 | 1.9 E-4 | 2.6 E-3 | 4.9 E-6 |
| 0.3 | 0.0886561 | 0.0889059 | 0.0797368 | 0.0886099 | 2.4 E-4 | 8.9 E-3 | 4.6 E-5 |
| 0.4 | 0.155767 | 0.155987 | 0.134773 | 0.155695 | 2.1 E-4 | 2.0 E-2 | 7.2 E-5 |
| 0.5 | 0.239713 | 0.239861 | 0.199079 | 0.239691 | 1.4 E-4 | 4.0 E-2 | 2.1 E-5 |
| 0.6 | 0.33785 | 0.333884 | 0.269344 | 0.338939 | 9.8 E-5 | 6.9 E-2 | 1.5 E-4 |
| 0.7 | 0.450952 | 0.451076 | 0.342124 | 0.451411 | 1.2 E-4 | 1.0 E-1 | 4.5 E-4 |
| 0.8 | 0.573885 | 0.574113 | 0.413894 | 0.574712 | 2.2 E-4 | 1.5 E-1 | 8.2 E-4 |
| 0.9 | 0.704994 | 0.705338 | 0.481116 | 0.706085 | 3.4 E-4 | 2.2 E-1 | 1.0 E-3 |
| 1.0 | 0.841471 | 0.841762 | 0.540302 | 0.842428 | 2.9 E-4 | 3.0 E-1 | 9.5 E-4 |



(a) Graph of exact and OHAM solutions





(b)Graph of exact and HPM solutions (c) Graph of exact and HAM solutions

Figure 3.1

**Problem 3.2:**

[Second order linear]

For $y \in [0,1]$ we consider the following differential equation

$$F''(y) + 2F'(y) + F(y) = 0, F(0) = 1, F'(0) = 0 \qquad (3.1c)$$

The exact solution of the problem is

$$F(y) = e^{-y} + ye^{-y} \qquad (3.2c)$$

Applying the technique, OHAM our zeroth order problem is

$$F_0''(y) + 2F_0'(y) + F_0(y) = 0, F_0(0) = 1, F_0'(0) = 0 \qquad (3.3c)$$

Its solution is

$$F_0(y) = \cos y \qquad (3.4c)$$

First order problem is

$$F_1''(y,C_1) = -F_1(y) - 2F_1'(y) + (1+C_1)[F_0(y) + 2F_0'(y) + F_0''(y)], F_1(0) = 0, F_1'(0) = 0 \qquad (3.5c)$$

Its solution is

$$F_1(y,C_1) = \frac{1}{2}\begin{pmatrix} 2y\cos y - \sin y + \cos 2y \sin y - \cos y \sin 2y + \\ 2y\cos y C_1 - \sin y C_1 + \cos 2y \sin y C_1 \\ -\cos y \sin 2y C_1 \end{pmatrix} \qquad (3.5c)$$

(3.6c)

Second order problem is

$$F_2''(y,C_1,C_2) = \begin{pmatrix} C_2F_0(y) - F_2(y) + 2C_2F_0'(y) - 2F_2'(y) + \\ C_2F_0''(y) + (1+C_1)[F_1(y) + 2F_1'(y) + F_1''(y)] \\ F_2(0) = 0, F_2'(0) = 0 \end{pmatrix}$$

(3.7c)

Its solution is

$$F_2(y,C_1,C_2) = \frac{1}{4}\begin{pmatrix} \cos y + 4y\cos y + 2y^2\cos y - \\ \cos y \cos 2y - 2\sin y + 2\cos 2y \sin y + \\ 2y\cos 2y \sin y - 2\cos y \sin 2y - \\ 2y\cos y \sin 2y - \sin y \sin 2y + \\ 2\cos y C_1 + 8y\cos y C_1 + 4y^2\cos y C_1 - \\ 2\cos y \cos 2y C_1 - 4\sin y C_1 + \\ 4\cos 2y \sin y C_1 + 4y\cos 2y \sin y C_1 - \\ 4\cos y \sin 2y C_1 - 4y\cos y \sin 2y C_1 - \\ 2\sin y \sin 2y C_1 + \cos y C_1^2 + 4y\cos y C_1^2 + \\ 2y^2\cos y C_1^2 - \cos y \cos 2y C_1^2 - \\ 2\sin y C_1^2 + 2\cos 2y \sin y C_1^2 + \\ 2y\cos 2y \sin y C_1^2 - 2\cos y \sin 2y C_1^2 \\ -2y\cos y \sin 2y C_1^2 - \sin y \sin 2y C_1^2 + \\ 4y\cos y C_2 - 2\sin y C_2 + 2\cos 2y \sin y C_2 - \\ 2\cos y \sin 2y C_2 \end{pmatrix}$$

(3.8c)

Third order problem is

$$F_0''(y,C_1,C_2,C_3)=\left(\begin{array}{l} C_3F_0(y)+C_2F_1(y)-F_3(y)+ \\ 2C_3F_0'(y)+2C_2F_1'(y)-2F_3'(y)+ \\ C_3F_0''(y)+C_2F_1''(y)(1+C_1)[F_2(y)+ \\ 2F_2'(y)+F_2''(y), \\ F_3(0)=0, F_3'(0)=0 \end{array}\right) \quad (3.9c)$$

Its solution is

$$F_3(y,C_1,C_2,C_3)=\frac{1}{2}\left(\begin{array}{l} C_2F_0(y)[-\cos y-2y\cos y+ \\ 2\cos^2 y-\cos y\cos 2y+\sin y- \\ 2y\sin y+2\cos^2 y\sin y- \\ \cos 2y\sin y+2\sin^2 y-\sin y\sin 2y]+ \\ C_1C_2F_0(y)[-\cos y-2y\cos y+2\cos^2 y- \\ \cos y\cos 2y+\sin y-2y\sin y+ \\ 2\cos^2 y\sin y-\cos 2y\sin y+2\sin^2 y- \\ \sin y\sin 2y]+C_3F_0(y)[-2\cos y+ \\ 2\cos^2 y+2\sin^2 y]+F_1(y)[-\cos y- \\ 2y\cos y+2\cos^2 y-\cos y\cos 2y+ \\ \sin y-2y\sin y+2\cos^2 y\sin y- \\ \cos 2y\sin y+2\sin^2 y-\sin y\sin 2y]+ \\ C_1F_1(y)[-2\cos y-4y\cos y+ \\ 4\cos^2 y+2\cos y\cos 2y+ \\ 2\sin y-4y\sin y+4\cos^2 y\sin y- \\ 2\cos 2y\sin y+4\sin^2 y-2\sin y\sin 2y]+ \\ C_1^2F_1(y)[-\cos y-2y\cos y+2\cos^2 y- \\ \cos y\cos 2y+\sin y-2y\sin y+ \\ 2\cos^2 y\sin y-\cos 2y\sin y+2\sin^2 y- \\ \sin y\sin 2y]+C_2F_1(y)[-2\cos y+ \\ 2\cos^2 y+2\sin^2 y] \end{array}\right)$$

(3.10c)

We use equations (3.4c), (3.6c), (3.8c) and (3.10c) we get third order approximate OHAM solution for $p=1$

$$\tilde{F}(y,C_1,C_2,C_3)=F_0(y)+F_1(y,C_1)+F_2(y,C_1,C_2)+F_3(y,C_1,C_2,C_3) \quad (3.11c)$$

Using the OHAM technique of section mentioned above on domain $a=0, b=1$ we use the residual $R$ that is

$$R=\tilde{F}''(y)+2\tilde{F}'(y)+\tilde{F}(y) \quad (3.12c)$$

We have obtained the following values of $C_1, C_2, C_3$.where

$$C_1=-0.6157891011374226, C_2=-1.648024704278988,$$
$$C_3=-0.48317914168990783 \quad (3.13c)$$

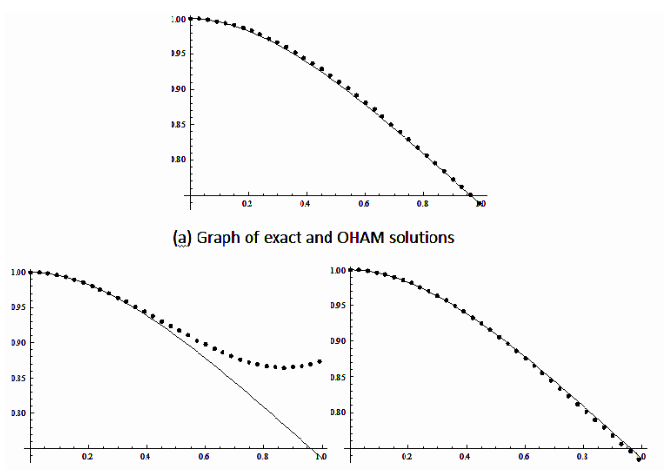
From the above values of $C_i$'s we get the following approximate solution

$$\tilde{F}(y)=1-0.424995y^2-0.161002y^3 \quad (3.14c)$$

The following table 3.2displays values of the exact solution (3.2c), OHAM solution (3.14c) and Error of OHAM. We compare the two solutions there exists similarity approximations between them also the values of HPM, HAM along with their errors are considered and the comparison is established between the errors of OHAM, HPM and HAM the errors of the technique, OHAM are smaller than the other two. All the mentioned values are described in the following table 3.2, their graphs are drawn in figure 3.2 below.

The following table 3.2 displays values of OHAM solution, exact solution and error

| $y$ | Exact sol | OHAM sol | HPM sol | HAM sol | Er OHAM | Er HPM | Er HAM |
|---|---|---|---|---|---|---|---|
| 0.0 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 0.0 E-0 | 0.0 E-0 | 0.0 E-0 |
| 0.1 | 0.995321 | 0.99591 | 0.995338 | 0.995562 | 5.9 E-4 | 1.6 E-5 | 2.4 E-4 |
| 0.2 | 0.982477 | 0.984288 | 0.982733 | 0.983077 | 1.8 E-3 | 2.5 E-5 | 5.9 E-4 |
| 0.3 | 0.963064 | 0.966098 | 0.964338 | 0.963763 | 3.0 E-3 | 1.2 E-3 | 6.9 E-4 |
| 0.4 | 0.938448 | 0.942305 | 0.9424 | 0.938806 | 3.8 E-3 | 3.9 E-3 | 3.5 E-4 |
| 0.5 | 0.909796 | 0.923877 | 0.919271 | 0.90935 | 4.0 E-3 | 9.4 E-3 | 4.4 E-4 |
| 0.6 | 0.878099 | 0.881778 | 0.8974 | 0.876489 | 3.6 E-3 | 1.9 E-2 | 1.6 E-3 |
| 0.7 | 0.844195 | 0.846976 | 0.879337 | 0.841258 | 2.7 E-3 | 3.5 E-2 | 2.9 E-3 |
| 0.8 | 0.808792 | 0.810436 | 0.867733 | 0.804626 | 1.6 E-3 | 5.8 E-2 | 4.1 E-3 |
| 0.9 | 0,772482 | 0.773125 | 0.865338 | 0.76749 | 6.4 E-4 | 9.2 E-2 | 4.9 E-3 |
| 1.0 | 0.735759 | 0.736007 | 0.875 | 0.73066 | 2.4 E-4 | 1.3 E-1 | 5.0 E-3 |



(a) Graph of exact and OHAM solutions

(b) Graph of exact and HPM solution (c) Graph of exact and HAM solutions

Figure 3.2

**Problem 3.3:**

For $y\in[0,1]$ we consider the following linear differential equation

$$F'''(y)+F(y)-(7-y^2)\cos y-(y^2-6y-1)\sin y=0,$$
$$F(0)=0, F'(0)=-1, F'(1)=2\sin(1) \quad (3.1d)$$

The exact solution of the problem is [third order linear]

$$F(y) = (y^2 - 1)\sin y \qquad (3.2d)$$

Applying the technique of OHAM that is described in the above section. The zeroth order problem is

$$F_0'''(y) = 0, F_0(0) = 0, F_0'(0) = -1, F_1'(1) = 2\sin(1) \qquad (3.3d)$$

Its solution is

$$F_0(y) = \frac{1}{2}\left(-2y + y^2(1 + 2\sin(1))\right) \qquad (3.4d)$$

First order problem is

$$F_1'''(y, C_1) = \begin{pmatrix} C_1[-7\cos y + y^2\cos y + \sin y + 6y\sin y - \\ y^2\sin y + F_0(y)] + (1 + C_1)F_0'''(y) \\ F_1(0) = 0, F_1'(0) = 0, F_1'(1) = 0 \end{pmatrix} \qquad (3.5d)$$

Its solution is

$$F_1(y, C_1) = \frac{1}{240}C_1\begin{pmatrix} -3120 - 240y + 135y^2 - 10y^4 + \\ 2y^5 - 480y^2\cos(1) + 3120\cos y - \\ 240y^2\cos y + 950y^2\sin(1) + \\ 4y^5\sin(1) + 240\sin y + \\ 1440y\sin y - 240y^2\sin y \end{pmatrix} \qquad (3.6d)$$

Second order problem is

$$F_2'''(y, C_1, C_2) = \begin{pmatrix} C_2[-7\cos y + y^2\cos y + \sin y + \\ 6y\sin y - y^2\sin y + F_0(y) + F_0'''(y)] + \\ C_1F_1(y) + (1 + C_1)F_1'''(y), \\ F_2(0) = 0, F_2'(0) = 0, F_2'(1) = 0 \end{pmatrix} \qquad (3.7d)$$

Its solution is

$$F_2(y, C_1, C_2) = \frac{1}{40320}\begin{pmatrix} C_1[-52416 - 40320y + 22680y^2 - 1680y^4 \\ +336y^5 - 80640y^2\cos(1) + 524160\cos y - \\ 40320y^2\cos y + 159600y^2\sin(1) + \\ 672y^5\sin(1) + 40320\sin y + \\ 241920y\sin y - 40320y^2\sin y] + \\ C_1^2[-1048320 + 1209600y - \\ 468801y^2 - 87360y^3 - 3360y^4 + \\ 714y^5 - 8y^7 + y^8 + 446880y^2\cos(1) - \\ 1344y^5\cos(1) + 1048320\cos y + \\ 483840y\cos y - 80640y^2\cos y + \\ 475502y^2\sin(1) + 3332y^5\sin(1) + \\ 2y^8\sin(1) - 1693440\sin y + \\ 483840y\sin y] + C_2[-524160 - \\ 40320y + 22680y^2 - 1680y^4 + \\ 336y^5 - 80640y^2\cos(1) + \\ 524160\cos y - 40320y^2\cos y + \\ 159600y^2\sin(1) + 672y^5\sin(1) + \\ 40320\sin y + 241920y\sin y - \\ 40320y^2\sin y] \end{pmatrix} \qquad (3.8d)$$

Third order problem is

$$F_3'''(y, C_1, C_2, C_3) = \begin{pmatrix} C_3[-7\cos y + y^2\cos y + \sin y + \\ 6y\sin y - y^2\sin y + F_0(y) + \\ F_0'''(y)] + C_2[F_1(y) + F_1'''(y)] + \\ C_1F_2(y) + (1 + C_1)F_2'''(y) \\ F_3(0) = 0, F_3'(0) = 0, F_3'(1) = 0 \end{pmatrix} \qquad (3.9d)$$

Its solution is obtained by using the method ,OHAM .The optimal values of the auxiliary constants $C_1, C_2, C_3$ are obtained by using Galerkin's method or least square method, using these values we get the series

$$
F_3(y,C_1,C_2,C_3)=\frac{1}{159667200}\left(\begin{array}{l}
C_1[-2075673600-159667200y+\\
89812800y^2-6652800y^4+1330560y^5\\
-319334400y^2\cos(1)+2075673600\cos y-\\
159667200y^2\cos y+632016000y^2\sin(1)+\\
2661120y^5\sin(1)+159667200\sin y+\\
958003200y\sin y-159667200y^2\sin y]+\\
C_1^{\,2}[-8302694400+9580032000y-\\
3712903920y^2-691891200y^3-\\
26611200y^4+5654880y^5-63360y^7+\\
7920y^8+3539289600y^2\cos(1)-\\
10644480y^5\cos(1)+8302694400\cos y+\\
3832012800y\cos y-638668800y^2\cos y+\\
3765975840y^2\sin(1)+26389440y^5\sin(1)+\\
15840y^8\sin(1)-13412044800\sin y+\\
3832012800y\sin y]+C_1^{\,3}[8302694400+\\
14689382400y-6081277257y^2-\\
1037836800y^3+186278400y^4-\\
28113426y^5-2882880y^6-\\
95040y^7+12375y^8-44y^{10}+4y^{11}+\\
7443992160y^2\cos(1)+24171840y^5\cos(1)-\\
15840y^8\cos(1)-8302694400\cos y+\\
5748019200y\cos y-319334400y^2\cos y-\\
1867972634y^2\sin(1)+44577852y^5\sin(1)+\\
47190y^8\sin(1)+8y^{11}\sin(1)-\\
20437401600\sin y+319334400y^2\sin y]+\\
C_2[-2075673600+89812800y^2-\\
6652800y^4+1330560y^5-319334400y^2\cos(1)+\\
2075673600\cos y-159667200y^2\cos y+\\
632016000y^2\sin(1)+2661120y^5\sin(1)+\\
159667200\sin y+958003200y\sin y-\\
159667200y^2\sin y]+C_1C_2[-8302694400+\\
9580032000y-3712903920y^2-\\
691891200y^3-26611200y^4+565880y^5-\\
63360y^7+7920y^8+3539289600y^2\cos(1)-\\
10644480y^5\cos(1)+8302694400\cos y+\\
3832012800y\cos y-638668800y^2\cos y+\\
3765975840y^2\sin(1)+26389440y^5\sin(1)+\\
15840y^8\sin(1)-13412044800\sin y+\\
3822012800y\sin y]+C_3[-2075673600-\\
159667200y+89812800y^2-6652800y^4+\\
1330560y^5-319334400y^2\cos(1)+\\
2075673600\cos y+159667200y^2\cos y+\\
632016000y^2\sin(1)+2661120y^5\sin(1)+\\
159667200\sin y+958003200y\sin y-\\
159667200y^2\sin y]
\end{array}\right)
$$

Now we use equations (3.4d), (3.6d), (3.8d) and (3.10d) to get third order approximate solution by OHAM for $p=1$ that is

$$
\tilde{F}(y,C_1,C_2,C_3)=F_0(y)+F_1(y,C_1)+F_2(y,C_1,C_2)+F_3(y,C_1,C_2,C_3) \qquad (3.11d)
$$

Using the proposed technique of section described above on the domain $a=0,b=1$ we use the residual

$$
R=\tilde{F}'''(y)+\tilde{F}(y)-(7-y^2)\cos y-(y^2-6y-1)\sin y \qquad (3.12d)
$$

The following values of $C_i$'s are found

$C_1=1.062670102836802, C_2=-2.208103637790291,$
$C_3=0.33369993779942651$

We use the above values of $C_1,C_2,C_3$, the approximate solution is

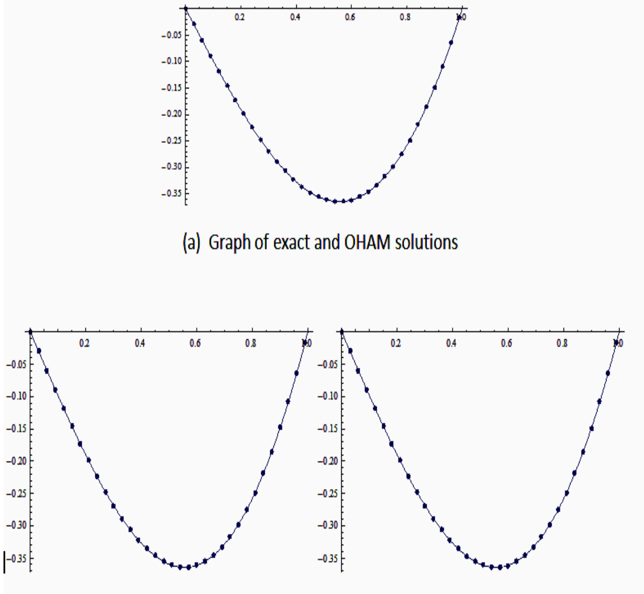
$$
\begin{aligned}
\tilde{F}(y)=&-y+0.0000488077y^2+1.16623y^3-\\
&0.173884y^5-0.00092637y^6+\\
&0.00852856y^7+0.000200624y^8-\\
&0.000245363y^9-1.12908\times10^{-6}y^{10}+\\
&4.06776\times10^{-6}y^{11}-3.01577\times10^{-9}y^{12}-\\
&4.10508\times10^{-8}y^{13}+1.67752\times10^{-10}y^{14}+\\
&2.89409\times10^{-10}y^{15}+O(y^{16})
\end{aligned} \qquad (3.13d)
$$

The following table 3.3 displays values of OHAM solution, exact solution and error

| y | Exact sol | OHAM sol | HPM sol | HAM sol | Er OHAM | Er HPM | Er HAM |
|---|---|---|---|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.0 E-0 | 0.0 E-0 | 0.0 E-0 |
| 0.1 | -0.0988351 | -0.0886448 | -0.0988249 | -0.0988314 | 1.9 E-4 | 1.0 E-5 | 3.6 E-6 |
| 0.2 | -0.190723 | -0.190034 | -0.190682 | -0.19071 | 6.8 E-4 | 4.0 E-5 | 1.2 E-5 |
| 0.3 | -0.268923 | -0.267535 | -0.268833 | -0.268899 | 1.3 E-3 | 9.0 E-5 | 2.4 E-5 |
| 0.4 | -0.327111 | -0.32492 | -0.326954 | -0.327074 | 2.1 E-3 | 1.5 E-4 | 3.7 E-5 |
| 0.5 | -0.359569 | -0.357602 | -0.35933 | -0.35952 | 3.0 E-3 | 2.3 E-4 | 4.9 E-5 |
| 0.6 | -0.361371 | -0.357602 | -0.361042 | -0.36131 | 3.7 E-3 | 3.2 E-4 | 6.0 E-5 |
| 0.7 | -0.328551 | -0.324145 | -0.328131 | -0.32848 | 4.4 E-3 | 4.2 E-4 | 7.1 E-5 |
| 0.8 | -0.258248 | -0.253368 | -0.257746 | -0.258169 | 4.8 E-3 | 5.0 E-4 | 7.9 E-5 |
| 0.9 | -0.148832 | -0.143667 | -0.148271 | -0.148747 | 5.1 E-3 | 5.6 E-4 | 8.4 E-5 |
| 1.0 | 0.000000 | 0.0052575 | 0.00058439 | 0.0000869 | 5.2 E-3 | 5.8 E-4 | 8.6 E-5 |

Table 3.3

Also in the above Table 3.3 the values of HPM and HAM solutions along with their errors are displayed and we conclude that the errors of technique, OHAM are smaller than HPM and HAM solutions. From the above table3.3 we conclude that OHAM and exact solutions are in best agreement and the values of the two columns are nearly equal.




(a) Graph of exact and OHAM solutions

(b) Graph of HPM and exact solutions (c) Graph of HAM and exact solutions

Figure 3.3

In figure 3.3(a), (b) and (c) we investigate that the graphs of OHAM, HPM and HAM are coincident with the Graphs of their exact solutions, but generally we conclude that the technique, OHAM is more effective than the other two. From the above figure 3.3we investigate that two graphs that is exact and OHAM solution graphs are coincident, this shows that the method OHAM is effective and reliable. Solid curve= exact solution, Dotted curve OHAM solution, HPM solution and HAM solution.

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