HERMITE HADAMARD INEQUALITY FOR ρ-CONVEX FUNCTIONS IN THE SECOND SENSE FOR FUZZY INTEGRALS

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ABSTRACT: *In this pape,r we prove a Hermite–Hadamard type inequality for ρ-convex functions in the second sense for fuzzy integrals. Some examples are given to manifest the results.*

Keywords: ρ-Hermite{Hadamard inequality, Sugeno integral, ρ-convex function.

INTRODUCTION

In 1974, Sugeno [16] inaugurated the thought of fuzzy measures and fuzzy integral as a gadget for modeling non-deterministic dilemmas. The theory of fuzzy integrals attracted the attention of many mathematicians and therefore the equities and utilizations of the Sugeno integral have been delibrated by bountiful authors. Ralescu and Adams [11] gave considerably commensurate interpretations of fuzzy integrals, Román-Flores et al. [12, 13] studied the level-continuity of fuzzy integrals and H-continuity of fuzzy measures and Wang and Klir [18] has given a generic analysis on fuzzy measurement and fuzzy integration theory. Freshly, Román-Flores et al. [14, 15] and Flores-Franulič et al. [6] presented some fuzzy integral inequalities.

In this paper we substantiate a Hermite-Hadamard type inequality for the Sugeno integral for functions which are ρ-convex in the second sense.

For a seek to progress our results we first give some elemental characters and properties about Sugeno integral. For elaboration on Sugeno integral we invoke the readers to [16, 18].

Suppose that Σ is σ-algebra of subsets of $\mathbb{R}$ and that $\phi:\Sigma\to[0,\infty)$, is non-negative, extended real valued set function, then η aforesaid to be fuzzy measure contingent upon:

1. $\eta(\emptyset)=0$,
2. $C,\ G\in\Sigma$ and $C\subset G$ imply that $\eta(C)\le\eta(G)$ (monotonicity),
3. $\{C_r\}\subset\Sigma,\ C_1\subset C_2\subset\ldots$, imply $\lim_{r\to\infty}\eta(C_r)=\eta(\bigcup_{r=1}^{\infty}C_r)$ (continuity from below) and
4. $\{C_r\}\subset\Sigma,\ C_1\supset C_2\supset\ldots,\ \eta(C_1)<\infty$, imply $\lim_{r\to\infty}\eta(C_r)=\eta(\bigcap_{r=1}^{\infty}C_r)$ (continuity from above).

If s is a non-negative real-valued function defined on $\mathbb{R}$, we will denote by $L_\zeta s=\{x\in\mathbb{R}:s(x)\ge\zeta\}=\{s\ge\zeta\}$ the ζ-level of s, for $\zeta>0$ and $L_0 s=\{x\in\mathbb{R}:s(x)\ge 0\}=supps$ is the support of s. Observe that if $\zeta\le\beta$, then $\{s\le\zeta\}\subset\{s\le\beta\}$. If μ is fuzzy measure on $(\mathbb{R},\Sigma)$, by $\mathcal{G}^{\mu}(\mathbb{R})$, we denote all η-measurable functions from $\mathbb{R}$ to $[0,\infty)$.

Suppose that η is a fuzzy measure on $(\mathbb{R},\Sigma)$. If $s\in\mathcal{G}^{\mu}(\mathbb{R})$ and $A\subset\Sigma$ then the Sugeno integral (or fuzzy integral) of s on A with respect to the fuzzy measure μ is defined as

$$\int_A s d\eta=\bigvee_{\zeta\ge0}\,[\zeta\wedge\eta(A\cap\{s\ge\zeta\})],$$

where $\vee$ and $\wedge$ denote the operations sup and inf on $[0,\infty)$, respectively. The following properties of the Sugeno integral are well known and can be found in [18].

Proposition 1 *η is a fuzzy measure on $(\mathbb{R},\Sigma)$, $A\subset\Sigma$ and $s,t\in\mathcal{G}^{\eta}(\mathbb{R})$ then*

1. $\int_A s d\eta\le\eta(A)$.
2. $\int_A k d\eta=k\wedge\eta(A)$.
3. If $s\le t$ on A then $\int_A s d\eta\le\int_A t d\eta$.
4. $\eta(A\cap\{s\ge\zeta\})\ge\alpha\Rightarrow\int_A s d\eta\ge\zeta$.
5. $\eta(A\cap\{s\ge\zeta\})\le\alpha\Rightarrow\int_A s d\eta\le\zeta$.
6. $\int_A s d\eta<\alpha\Leftrightarrow$ there exists $\gamma<\zeta$ such that $\eta(A\cap\{s\ge\gamma\})<\zeta$.
7. $\int_A s d\eta>\alpha\Leftrightarrow$ there exists $\gamma>\zeta$ such that $\eta(A\cap\{s\ge\gamma\})>\zeta$.

Remark 1 *Consider the distribution function G associated to s on A, that is, $G(\zeta)=\eta(A\cap\{s\ge\gamma\})$. Then from (4) and (5) of Proposition 1, we have that*

$$G(\zeta)=\zeta\Rightarrow\int_A s d\eta=\zeta.$$

Therefore it follows that any fuzzy integral can be calculated by solving the equation $G(\zeta)=\zeta$.

2 ρ-Hermite-Hadamard Inequality Considering Fuzzy Integrals

In [4], J. Caballero, K. Sadarangani has proven with the help of certain examples that the classical Hermite–Hadamard inequalities (see [7, 8] for the history of these inequalities):

$$s\left(\frac{d+e}{2}\right)\le\frac{1}{e-d}\int_d^e s(x)dx\le\frac{s(d)+s(e)}{2},\qquad(1)$$

where $s:[d,e]\to\mathbb{R}$ is a convex function, do not hold true for fuzzy integrals in general . In [4], the authors proved some Hermite-Hadamard type inequalities for fuzzy integrals and some examples were also given to illustrate their results .

In this section we aim to prove some Hermite-Hadamard type inequalities for functions which are ρ-convex in the second sense for fuzzy integrals.

In literature [2], a function $s:[0,\infty)\to[0,\infty)$ is an ρ-convex in the second sense, or that s belongs to the class K_ρ^2, if

$$s(\zeta x+(1-\zeta)y)\le\zeta^{\rho}s(x)+(1-\zeta)^{\rho}s(y),$$

holds for all x, $y\in[0,\infty)$, $\zeta\in[0,1]$, for some fixed $\rho\in(0,1]$.

In [5], Dragomir and Fitzpatrick proved a variant of Hadamard's inequalities which hold for s-convex functions in the second sense:

$$2^{\rho-1}s\left(\frac{d+e}{2}\right)\le\frac{1}{e-d}\int_d^e s(x)dx\le\frac{s(d)+s(e)}{\rho+1},\qquad(2)$$

where (2) is known in literature as ρ-Hermite-Hadamard inequalities. For more about the properties on ρ-convex functions and ρ-Hermite-Hadamard inequalities we refer the interested readers to [1, 2, 3, 5, 9].

Unfortunately, as we will see, the ρ-Hermite-Hadamard inequalities do not valid as well for fuzzy integrals in general. We quote below a very important example of s-convex functions from [5] to be used in the sequel:

Let $\rho \in (0,1)$ and $d, e, c \in \mathbb{R}$. We define a function $s: [0,\infty) \to \mathbb{R}$, by

$$s(n) = \begin{cases} d & n = 0, \\ en^{\rho} + c & n > 0. \end{cases}$$

If $e \geq 0$ and $0 \leq c \leq e$, then $s \in K_{\rho}^2$. Hence, for $d = c = 0$, $e = 1$, we have $s: [0, e^*] \to [0, e^*]$, $s(n) = n^{\rho}$, $0 < \rho < 1$, $s \in K_{\rho}^2$, $e^* > 0$ (see e.g.[5]).

Example 1 *Consider $X = [0,1]$ and let η be the usual Lebesgue measure on X. If we take the function $s(x) = \sqrt{x}$, $x \in [0,1]$, then $s \in K_{\rho}^2$. Now we calculate the the Sugeno integral $\int_0^1 \sqrt{x} d\eta$ by using Remark 1. Consider the distribution function G associated to s on $[0,1]$, that is*

$$G(\zeta) = \eta([0,1] \cap \{s \geq \zeta\}) = \eta([0,1] \cap \{\sqrt{x} \geq \zeta\})$$
$$= \eta([0,1] \cap \{x \geq \zeta^2\}) = 1 - \zeta^2$$

and we solve the equation $1 - \zeta^2 = \zeta$, we obtain that $\zeta = \frac{-1+\sqrt{5}}{2}$. By Remark 1 we have

$$\int_0^1 \sqrt{x} d\eta = \frac{-1+\sqrt{5}}{2} \approx 0.618033$$

on the other hand for $\rho = \frac{7}{8}$, we have

$$2^{\frac{-1}{8}} s\left(\frac{1}{2}\right) = 2^{\frac{-1}{8}} \times \frac{1}{\sqrt{2}} \approx 0.64842.$$

This shows that the left part of the ρ-Hermite-Hadamard inequality is not valid in the fuzzy context in general.

Example 2 *Consider $X = [0,1]$ and let η be the usual Lebesgue measure on X. If we take the function $s(x) = \frac{\sqrt{x}}{3}$, then $s \in K_{\rho}^2$. Now again we calculate the the Sugeno integral $\int_0^1 \frac{\sqrt{x}}{3} d\eta$ by using Remark 1. Consider the distribution function G associated to s on $[0,1]$, that is*

$$G(\zeta) = \eta([0,1] \cap \{s \geq \zeta\})$$
$$= \eta([0,1] \cap \{\sqrt{x}/3 \geq \zeta\}) = \eta([0,1] \cap \{x \geq 9\zeta^2\})$$
$$= 1 - 9\zeta^2$$

and we solve the equation $1 - 9\zeta^2 = \zeta$, we obtain that $\zeta = \frac{-1+\sqrt{37}}{18}$. By Remark 1 we have

$$\int_0^1 \frac{\sqrt{x}}{2} d\eta = \frac{-1+\sqrt{37}}{18} \approx 0.2823756,$$

but on the other hand for $\rho = \frac{1}{2}$, we have

$$\frac{2[s(1)+s(0)]}{3} = \frac{2}{9} \approx 0.22222222.$$

Thus the right side of the ρ-Hermite-Hadamard inequality is not satisfied for fuzzy integrals. Now we present ρ-Hermite-Hadamard inequalities for the Sugeno integral.

Lemma 1 *Let $0 < p \leq 1$ and $x \geq 0$, $y \geq 0$. Then*

$$(x+y)^p \leq x^p + y^p.$$

Proof. When $p = 1$, then the result is obviously true. So assume that $0 < p < 1$ and consider the function

$$s(n) = 1 + n^p - (1+n)^p, n \geq 0.$$

Then $$s'(n) = pn^{p-1} - p(1+n)^{p-1}, n \geq 0.$$

Since $p - 1 < 0$, hence $s'(n) \geq 0$, $n \geq 0$. Thus

$$s(n) \geq 0, n \geq 0.$$

That is $(1+n)^p \leq n^p + 1, n \geq 0.$ (3)

If $y = 0$ then $(x+y)^p \leq x^p + y^p$ is true with equality sign, so let $y > 0$ and take $n = \frac{x}{y}$ in (2.3), we have

$$(1+y)^p \leq \left(\frac{x}{y}\right)^p + 1, y > 0$$

so that $(x+y)^p \leq x^p + y^p$.

This completes the proof.

Theorem 1 *$g: [0,1] \to (0,\infty]$ be an ρ-convex function in the second sense $\rho \in (0,1)$ and μ the Lebesgue measure on $\mathbb{R}$. Then*

$$\int_0^1 t d\eta \leq \min\{1, \zeta_1\},$$

where ζ_1 is a positive real solution of the equation $1 - \left(\frac{\zeta - t(0)}{t(0)+t(1)}\right)^{\frac{1}{\rho}} = \zeta$.

Proof. Since t is an ρ-convex function in the second sense $\rho \in (0,1)$, therefore for $x \in [0,1]$ we have

$t(x) = t((1-x)0 + 1 \cdot x) \leq (1-x)^{\rho} t(0) + x^{\rho} t(1)$ (4)

We observe that $1 - x \geq 0$, therefore we have that

$$1 - x = |1 - x| \leq 1 + x.$$

and hence by Lemma1, we have $(1-x)^{\rho} \leq (1+x)^{\rho} \leq 1 + x^{\rho}$.

Thus (2.4) becomes $t(x) \leq (1 + x^{\rho})t(0) + x^{\rho}t(1) = h(x)$.

By (3) of Proposition 1, we have that

$\int_0^1 t d\eta \leq \int_0^1 ((1+x^{\rho})t(0) + x^{\rho}t(1)) d\eta = \int_0^1 h(x) d\eta$. (5)

In order to calculate the integral in the right-hand part of the last inequality, we consider the distribution function G given by

$G(\zeta) = \eta([0,1] \cap \{h \geq \zeta\}) = \eta([0,1] \cap \{(1+x^{\rho})t(0) + x^{\rho}t(1) \geq \zeta\})$. (6)

Thus from (6) we get that

$$G(\zeta) = \eta([0,1] \cap \{h \geq \zeta\})$$
$$= \eta\left([0,1] \cap \left\{x \geq \left(\frac{\zeta - t(0)}{t(0)+t(1)}\right)^{\frac{1}{\rho}}\right\}\right)$$
$$= 1 - \left(\frac{\zeta - t(0)}{t(0)+t(1)}\right)^{\frac{1}{\rho}}.$$

Therefore we have $1 - \left(\frac{\zeta - t(0)}{t(0)+t(1)}\right)^{\frac{1}{\rho}} = \zeta$ (7)

From (1) of Proposition 1 we get that

$\int_0^1 h(x) d\eta \leq \eta([0,1]) = 1$ (8)

By Remark 1, (5) and (2.8), we have $\int_0^1 t d\eta \leq \min\{1, \zeta_1\}$, where ζ_1 is a positive real solution of (7). This completes the proof of the thoerem.

Corollary 1 *$g: [0,1] \to (0,\infty]$ be an ρ-convex function in the second sense $\rho \in (0,1)$ and η the Lebesgue measure on $\mathbb{R}$. If $t(0) = t(1) \neq 0$, then*

$$\int_0^1 t d\eta \leq \min\{1, \zeta_1\},$$

where ζ_1 is a positive solution of the equation

$$1 - \left(\frac{\zeta - t(0)}{2t(0)}\right)^{\frac{1}{\rho}} = \zeta.$$

Proof. It is direct consequence of the above theorem. Now we give examples to illustrate our result:

Example 3 *Consider the function $s(x) = x^{\frac{1}{3}}$ on $[0,1]$, then s is an ρ-convex function. Moreover $s(0) = 0$ and $s(1) = 1$, thus in particular for $\rho = \frac{1}{3}$, we have from the equation $1 - \left(\frac{\zeta - s(0)}{s(0)+s(1)}\right)^{\frac{1}{\rho}} = \zeta$, $\rho \in (0,1)$, which gives by solving by* *numerical methods, the positive real solution*

$$\alpha_1 = \sqrt[3]{\frac{1}{108}\sqrt{31}\sqrt{108}+\frac{1}{2}} - \frac{1}{3\sqrt[3]{\frac{1}{108}\sqrt{31}\sqrt{108}+\frac{1}{2}}} \approx 0.682\,33.$$

Therefore from Theorem 1 we have

$$\int_0^1 s d\eta \leq \sqrt[3]{\frac{1}{108}\sqrt{31}\sqrt{108}+\frac{1}{2}} - \frac{1}{3\sqrt[3]{\frac{1}{108}\sqrt{31}\sqrt{108}+\frac{1}{2}}}.$$

Note that one can choose other values of $\rho \in (0,1)$ to get another estimates for the integral $\int_0^1 x^{\frac{1}{3}} d\eta$.

Example 4 *Consider the function* $s(x) = x^{\frac{3}{4}}$ *on* $[0,1]$, *then* s *is an* ρ*-convex function. Moreover* $s(0) = 0$ *and* $s(1) = 1$, *thus in particular for* $\rho = \frac{3}{4}$, *we have from the equation* $1 - \left(\frac{\zeta - s(0)}{s(0)+s(1)}\right)^{\frac{4}{3}} = \zeta$, *which gives by solving by numerical methods, the positive real solution* $\zeta_1 \approx 0.549\,7$. *Therefore from Theorem 1 we have*

$$\int_0^1 s d\eta \leq 0.549\,7.$$

Note that one can choose other values of $\rho \in (0,1)$ to get another estimates for the integral $\int_0^1 x^{\frac{3}{4}} d\eta$.

Now we prove general case of Theorem 1 as follow:

Theorem 2 $t: [d,e] \to (0,\infty]$ *be an* ρ*-convex function in the second sense* $s \in (0,1)$, $0 \leq d < e < \infty$ *and* η *the Lebesgue measure on* $\mathbb{R}$, *then*

$$\int_d^e t d\eta \leq \min\{e-d, \zeta_1\},$$

where ζ_1 is a positive real solution of the equation

$$(e-d)\left[1 - \left(\frac{\zeta - t(d)}{t(d)+t(e)}\right)^{\frac{1}{\rho}}\right] = \zeta.$$

Proof. Since t is an ρ-convex function in the second sense $\rho \in (0,1)$, therefore for $x \in [d,e]$ we have

$$t(x) = t\left(\left(1 - \frac{x-d}{e-d}\right)d + \frac{x-d}{e-d}\cdot e\right)$$
$$\leq \left(1 - \frac{x-d}{e-d}\right)^{\rho} t(d) + \left(\frac{x-d}{e-d}\right)^{\rho} t(e) \quad (9)$$

Arguing similarly as in Theorem 1, we have that

$$1 - \frac{x-d}{e-d} = \left|1 - \frac{x-e}{d-e}\right| \leq 1 + \left|\frac{x-d}{e-d}\right| = 1 + \frac{x-d}{e-d}$$

and hence by Lemma1, we have

$$\left(1 - \frac{x-d}{e-d}\right)^{\rho} \leq \left(1 + \frac{x-d}{e-d}\right)^{\rho} \leq 1 + \left(\frac{x-d}{e-d}\right)^{\rho}.$$

Therefore (9) gives

$$t(x) \leq \left(1 + \left(\frac{x-d}{e-d}\right)^{\rho}\right) t(d) + \left(\frac{x-d}{e-d}\right)^{\rho} t(e) = h(x)$$

By (3) of Proposition 1, we have

$$\int_d^e t d\eta \leq \int_d^e \left[\left(1 + \left(\frac{x-d}{e-d}\right)^{\rho}\right) t(d) + \left(\frac{x-d}{e-d}\right)^{\rho} t(e)\right] d\eta = \int_d^e h d\eta \quad (10)$$

Let us consider the distribution function G associated to h on $[d,e]$ given by

$$G(\zeta) = \eta([d,e] \cap \{h \geq \zeta\})$$
$$= \eta\left([d,e] \cap \left\{\left(1 + \left(\frac{x-d}{e-d}\right)^{\rho}\right) t(d) + \left(\frac{x-d}{e-d}\right)^{\rho} t(e) \geq \zeta\right\}\right)$$
$$= \eta\left([d,e] \cap \left\{x \geq d + (e-d)\left(\frac{\zeta - t(d)}{t(d)+t(e)}\right)^{\frac{1}{\rho}}\right\}\right)$$
$$= (e-d)\left[1 - \left(\frac{\zeta - t(d)}{t(d)+t(e)}\right)^{\frac{1}{\rho}}\right]. \quad (11)$$

Therefore from (11), we have the equation

$$(e-d)\left[1 - \left(\frac{\zeta - t(d)}{t(d)+t(e)}\right)^{\frac{1}{\rho}}\right] = \zeta. \quad (12)$$

By (1) of Proposition 1 we get that

$$\int_d^e h(x) d\eta \leq \eta([d,e]) = e-d. \quad (13)$$

Thus by Remark 1, (10) and (13), we have

$$\int_d^e t d\eta \leq \min\{e-d, \zeta_1\},$$

where ζ_1 is a positive real solution of (13).

Corollary 2 $t: [d,e] \to (0,\infty]$ *be an* ρ*-convex function in the second sense* $\rho \in (0,1)$, $0 \leq d < e < \infty$ *and* η *the Lebesgue measure on* $\mathbb{R}$. *If* $t(d) = t(e) \neq 0$, *then*

$$\int_0^1 t d\eta \leq \min\{e-d, \zeta_1\},$$

where ζ_1 is a positive solution of the equation

$$(e-d)\left[1 - \left(\frac{\zeta - t(d)}{2t(d)}\right)^{\frac{1}{\rho}}\right] = \zeta.$$

Proof. It is direct consequence of the above theorem.

Example 5 *Let* $s(x) = 2x^{\frac{1}{3}}$ *be a function defined on* $[0,2]$, *then* s *is an* ρ*-convex function in the second sense . Here we have* $d = 0$, $e = 2$, *moreover* $s(d) = s(0) = 0$ *and* $s(e) = s(2) = 2^{\frac{4}{3}}$, *therefore by Theorem 2, we solve the equation* $(e-d)\left[1 - \left(\frac{\zeta - t(d)}{t(d)+t(e)}\right)^{\frac{1}{\rho}}\right] = \zeta$, *that is in particular for* $\rho = \frac{1}{3}$, *we solve the equation* $2\left[1 - \frac{\zeta^3}{16}\right] = \zeta$, *we get that*

$$\zeta_1 = \sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8} - \frac{8}{3\sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8}}$$
$$\approx 1.541\,8.$$

Hence we get the following estimate:

$$\int_0^2 s d\eta \leq \min\left\{2, \sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8} - \frac{8}{3\sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8}}\right\}$$
$$= \sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8} - \frac{8}{3\sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8}}.$$

Example 6 *Let* $s(x) = 2x^{\frac{1}{\sqrt{3}}}$ *be a function defined on* $[0,4]$, *then* s *is an* ρ*-convex function in the second sense. Here we have* $d = 0$, $e = 4$, *moreover* $s(d) = s(0) = 0$ *and* $s(e) = f(4) = 2\cdot 4^{\frac{1}{\sqrt{3}}}$, *therefore by Theorem 2, we solve the equation* $(e-d)\left[1 - \left(\frac{\zeta - t(d)}{t(d)+t(e)}\right)^{\frac{1}{\rho}}\right] = \zeta$, *that is in particular for* $\rho = \frac{1}{\sqrt{3}}$, *we solve the equation* $4\left[1 - \left(\frac{\zeta}{2\cdot 4^{\frac{1}{\sqrt{3}}}}\right)^{\sqrt{3}}\right] = \zeta$, *we* *get that* $\zeta \approx 2.513\,9$. *Hence we get the following estimate:*

$$\int_0^2 s d\eta \leq \min\{4, 2.5139\} = 2.5139.$$

Remark 2 *In the last two examples one can get different estimates for the integrals for different choices of* $\rho \in (0,1)$.

CONCLUSION

Hermite–Hadamard type inequality for ρ-convex functions in the second sense for fuzzy integrals is established, and our results are evident by examples .

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