BEST PROXIMITY POINT RESULT OF QUASI CONTRACTION IN CONE METRIC SPACES

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ABSTRACT*: The aim of this paper is to present best proximity point result of quasi contraction mappings in the frame work of regular cone metric spaces. Our result extend and generalize various results in the existing literature.*

Keywords: Cone metric spaces; best proximity point; quasi contraction

1.0 INTRODUCTION

The concept of cone metric spaces was introduced by Huang and Zhang, as a generalization of an ordinary metric spaces. They studied the convergence of sequences among other topological properties of such spaces, and then proved some fixed point results of contractive mappings in these spaces [12]. They gave the cone metric space version of Banach contraction principle and other related existing results in metric spaces. Later on, relaxing an assumption of normality of order inducing cone, Rezapour and Hagi [33] obtained some fixed point theorems, as the generalizations of the results in [12]. They presented a number of examples of non-normal cones to show that such generalizations are meaningful. Since then, many authors have been in the area of fixed point theory in the setting of cone metric spaces [1, 2, 5, 8, 10, 13, 14, 15,16, 17, 32, 33]).

On the other hand, Ćirić [6] introduced the notion of quasi-contraction as one of the most general contractive type maps and prove the related fixed point theorem. Ilić [13] obtained some fixed point resuts of quasi contraction on a cone (normal) metric space and hence generalized the results of Huang and Zhang [12] and Ćirić [6]. Kadelburg et al. [16] obtained a fixed point result of quasi contraction mappings without using the normality condition. For more work in this direction, we refer to [10] and references therein.

Kirk *et al*. [21] generalized the Banach contraction principle employing the concept of cyclic contraction mappings on two closed subsets of a complete metric space. Petrusel [26] proved some results about periodic points of cyclic contraction maps and generalized the main result in [21]. Eldered and Veeramani [9] proved some results about best proximity points of cyclic contraction maps.

Recently, Hagi et al. [11] defined the notion of distance between two subsets in regular cone metric spaces and studied some necessary conditions to obtain the existance of best proximity points of cyclic contraction mappings on regular cone metric spaces. Abbas et al. [4] introduced the notions of proximal cyclic contractions and prove best proximity point results in the setup of regular cone metric space. For more results on best proximity points we refer to [3],18-20, 24-25, 27-31,34-36] and references therein).

We further continue the work in this direction to prove the best proximity point results of quasi contraction mappings in regular cone metric spaces. Our results generalize the results in [10].

2.0 PRELIMINARIES

Following definitions and results will be needed in the sequel:
A subset P of a Banach space E is called a cone if

1. P is non-empty closed and $P \neq \{\theta\}$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P \cap (-P) = \{\theta\}$,

where θ denotes the zero element of the E. For a given cone $P \subset E$, we can define a partial ordering $\leq$ with respect to P as follows: $x \leq y$ if and only if $y - x \in P$. By $x < y$, we mean $x \leq y$ and $x \neq y$, while $x = y$ stands for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P. If $\mathrm{int}P \neq \phi$ then P is called a solid cone.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in P$,

$$\theta \leq x \leq y \Longrightarrow \| x \| \leq M \| y \|.$$

The least positive number satisfying above is called the normal constant of P [12].

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq ... \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \| x_n - x \| = 0$.

Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is known that every regular cone is normal [33].

In the following we assume that P is a cone in E with $\mathrm{int}P \neq \phi$ and $\leq$ is the partial ordering with respect to P.

Definition 2.1 (See [12], [23] and [26]) Let X be a non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies

1. $\theta \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$ for all $x, y \in X$;
3. $d(x,y) \leq d(x,z) + d(z,x)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space with a Banach space E.

Following are some known examples of cone metric space:

Example 2.2 Let $X = \mathrm{P}$, $E = \mathrm{P}^n$ and $P = \{(x_1,...,x_n) \in \mathrm{P}^n : x_i \geq 0\}$. It is easy to see that $d : X \times X \to E$ defined by

$$d(x,y) = (|x-y|, k_1|x-y|, ..., k_{n-1}|x-y|)$$

is a cone metric on X, where $k_i \geq 0$ for all $i \in \{1,...,n-1\}$.

Example 2.3 [33] Let $E = C^1[0,1]$ with

$\| x \| = \| x \|_\infty + \| x' \|_\infty$

on $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0,1]\}$. This cone is not normal. Consider, for example,

$$x_n(t) = \frac{1 - sinnt}{n+2} \text{ and } y_n(t) = \frac{1 + sinnt}{n+2}.$$

Since $\| x_n \| = \| y_n \| = 1$ and

$$\| x_n + y_n \| = \frac{2}{n+2} \to 0,$$

it follows that P is non-normal.

A sequence $\{x_n\}$ in a cone metric space X is called a convergent sequence if for every c in E with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ and $x \in X$ such that for all $n > n_0$, $d(x_n, x) = c$. We denote it by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$; $n \to \infty$. If for every c in E with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) = c$, then $\{x_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Let (X,d) be a cone metric space. Then we have the following properties:

(P1) If $u \leq v$ and $v = w$ then $u = w$.

(P2) If $\theta \leq u \ll c$ for each $c \in intP$ then $u = \theta$.

(P3) If $a \leq b + c$ for each $c \in intP$ then $a \leq b$.

(P4) If $\theta \leq x \leq y$, and $a \geq \theta$, then $\theta \leq ax \leq ay$.

(P5) If $\theta \leq x_n \leq y_n$ for each $n \in \mathrm{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $\theta \leq x \leq y$.

(P6) If $\theta \leq d(x_n, x) \leq b_n$ and $b_n \to \theta$, then $x_n \to x$.

(P7) If E is a real Banach space with a cone P and if $a \leq ka$, where $a \in P$ and $0 < k < 1$, then $a = \theta$.

(P8) If $c \in intP$, $\theta \leq a_n$ and $a_n \to \theta$, then there exists n_0 such that for all $n > n_0$ we have $a_n = c$.

From (P8), it follows that the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to \theta$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$.

Throughout this paper, E is a real Banach space, (X,d) is a regular cone metric space, $\leq$ is the partial ordering with respect to P and A, B are nonempty subsets of X. We say that A is bounded whenever there exists $e \gg \theta$ such that $d(x,y) \leq e$ for all $x, y \in A$.

Lemma 2.4 [7] Let (X,d) be a cone metric space. Suppose that $\{x_n\}$ is a sequence in X and that $\{b_n\}$ is sequence in E. If

$$\theta \leq d(x_n, x_m) \leq b_n$$

for $m > n$ and $b_n \to \theta$, $n \to \infty$, then x_n is a Cauchy sequence.

Definition 2.5 [11] An element $p \in P$ is said to be a lower bound for $A \times B$ whenever

$$p \leq d(a,b)$$

for all $(a,b) \in A \times B$. Moreover, if $p \leq q$ for all lower bound q for $A \times B$, then p is called the greatest lower bound for $A \times B$. In this case, we denote it by $dis(A,B)$. Clearly in above definition, $dis(A,B)$ is a unique vector in P. Also, θ is always a lower bound for $A \times B$.

3.0 MAIN RESULTS

In this section, we define the notion of quasi contraction for a non-self-mapping and prove best proximity point theorem in the framework of regular cone metric spaces.

If $p = d(A,B)$, we set

$$A_0 = \{x \in A : d(x, y) = p \text{ for some } y \in B\}$$

and

$$B_0 = \{y \in B : d(x, y) = p \text{ for some } x \in A\}.$$

We start with the following definition:

Definition 3.1 Let A and B be closed subsets of a cone metric space (X,d), and let $f : A \to B$. Then f is called quasi contraction if for some constant $\alpha \in (0,1)$ and for every $x, y \in A$, there exists an element

$$u \in C(f; x, y) = \{d(x,y) - p, d(x, fx) - p, d(y, fy) - p, d(x, fy) - p, d(y, fx) - p\},$$

such that

$$d(fx, fy) \leq \alpha \cdot u, \qquad (1)$$

for some $p \in P$.

Theorem 3.2 Let A, B be closed subsets of a complete cone metric space (X,d) with a solid cone P, $int(P) \neq \varphi$. Let $f : A \to B$ be continuous quasi-contraction with $f(A) \subseteq B$. Then f has unique best proximity point x^* in A.

Proof. First we will prove the following inequalities for arbitrary $x \in X$:

$$(i)\ d(f^n(x), f(x)) \leq \alpha(1-\alpha)^{-1} d(f(x), x) - p),$$

$(ii)\ d(f^n(x),x) \le (1-\alpha)^{-1}(d(f(x),x)-p),$ for $n \in \mathbb{N}$.

(i) is true for $n=1$. Suppose that it's satisfied for each $m \le n$. Since $d(f^{n+1}(x), f(x)) \le \alpha(u),$ where

$$u \in \{d(f^n(x),x)-p, d(f^n(x), f^{n+1}(x))-p, d(x, f(x))-p, d(f^n(x), f(x))-p, d(x, f^{n+1}(x))-p\},$$

we will consider the following cases:

Case-I: If $u = d(f^n(x),x)-p$, then

$$\begin{aligned} &d(f^{n+1}(x), f(x)) \\ &\le \alpha(d(f^n(x),x)-p) \\ &\le \alpha(d(f^n(x), f(x))) + \alpha(d(f(x),x)-p) \\ &\le \alpha(1-\alpha)^{-1}\alpha\cdot(d(f(x),x)-p) + \alpha(d(f(x),x)-p) \\ &= -\alpha(d(f(x),x)-p) + (1-\alpha)^{-1}\alpha(d(f(x),x)-p) \\ &\quad + \alpha(d(f(x),x)-p) \\ &= (1-\alpha)^{-1}\alpha(d(f(x),x)-p). \end{aligned}$$

$$\Rightarrow d(f^{n+1}(x), f(x)) \le (1-\alpha)^{-1}\alpha(d(f(x),x)-p).$$

Case-II: If $u = d(f^n(x), f(x))-p$, then

$$\begin{aligned} &d(f^{n+1}(x), f(x)) \\ &\le \alpha(d(f^n(x), f(x))-p) \\ &\le \alpha((1-\alpha)^{-1}\alpha(d(f(x),x)-p)-p) \\ &= (1-(1-\alpha))((1-\alpha)^{-1}\alpha\cdot(d(f(x),x)-p)-p) \\ &= (1-\alpha)^{-1}\alpha(d(f(x),x)-p) - \alpha\cdot(d(f(x),x)-p) \\ &\quad -\alpha\cdot(p) \\ &\le (1-\alpha)^{-1}\alpha(d(f(x),x)-p) \end{aligned}$$

$$\Rightarrow d(f^{n+1}(x), f(x)) \le (1-\alpha)^{-1}\alpha(d(f(x),x)-p).$$

Case-III: If $u = d(f(x),x)-p$, then

$$\begin{aligned} &d(f^{n+1}(x), f(x)) \\ &\le \alpha(d(f(x),x)-p) \le (1-\alpha)^{-1}\alpha(d(f(x),x)-p). \end{aligned}$$

$$\Rightarrow (f^{n+1}(x), f(x)) \le (1-\alpha)^{-1}\alpha(d(f(x),x)-p).$$

Case-IV: If $u = d(x, f^{n+1}(x))-p.$ Using the triangular inequality,

$$\begin{aligned} &d(f^{n+1}(x), f(x)) \\ &\le \alpha(d(x, f^{n+1}(x))-p) \\ &\le \alpha(d(x, f(x)) + d(f(x), f^{n+1}(x))-p), \\ &= \alpha(d(x, f(x))-p) + \alpha d(f(x), f^{n+1}(x)), \end{aligned}$$

hence,

$$d(f^{n+1}(x), f(x)) \le (1-\alpha)^{-1}\alpha(d(x, f(x))-p).$$

Case-V: If $u = d(f^n(x), f^{n+1}(x))-p$, then $d(f^{n+1}(x), f(x)) \le \alpha(d(f^n(x), f^{n+1}(x))-p)$ and since f is a quasi-contraction, we have

$$\begin{aligned} &d(f^n(x), f^{n+1}(x))-p \\ &\le \alpha^{n-1+i}(d(f(x), f^j(x))-p), \end{aligned}$$

for some $i \in \{0,1,2,...,n\}$, $j \in \{1,2,n+1\}$. The case where $j = n+1$, we have

$$d(f^{n+1}(x), f(x)) \le (1-\alpha^{n+i})^{-1}(\theta) = \theta$$

imples

$$d(f^{n+1}(x), f(x)) = \theta.$$

On the other hand

$$\begin{aligned} &d(f^{n+1}(x), f(x)) \\ &\le \alpha^{n+i}(d(f(x), f^j(x))-p) \\ &\le \alpha^{n+i}(1-\alpha)^{-1}\alpha(d(f(x),x)-p) \\ &\le (1-\alpha)^{-1}\alpha(d(f(x),x)-p), \end{aligned}$$

implies

$d(f^{n+1}(x), f(x)) \le (1-\alpha)^{-1}\alpha(d(f(x),x)-p).$ Hence, the inequality (i) holds for all $n \in \mathrm{N}$.

The inequality (ii) is obtained by (i) as:

$$\begin{aligned} &d(f^n(x),x) \le d(f^n(x), f(x)) + d(f(x),x) \\ &\le (1-\alpha)^{-1}\alpha(d(f(x),x)-p) + d(f(x),x) \\ &= (1-\alpha)^{-1}\alpha(d(f(x),x)) - (1-\alpha)^{-1}\alpha\cdot(p) \\ &\quad + d(f(x),x) \\ &= (1-\alpha)^{-1}(d(f(x),x)) - (1-\alpha)^{-1}\alpha(p) \\ &= (1-\alpha)^{-1}(d(f(x),x) - \alpha(p)) \end{aligned}$$

$n \in \mathrm{N}$. We now show that $\{f^n(x)\}$ is a Cauchy sequence in A. Suppose that $n,m \in \mathrm{N}$, $m > n$.

Since f is quasi contraction, so there exist $i, j \in \mathrm{N}$ with $1 \le i \le n$, $1 \le j \le m$,

$$\begin{aligned} &d(f^n(x), f^m(x)) \\ &\le \alpha^{n-1}(d(f^i(x), f^j(x))-p) \\ &\le \alpha^{n-1}(d(f^i(x), f(x)) + d(f(x), f^j(x))-p) \\ &\le \alpha^{n-1}((1-\alpha)^{-1}\alpha(d(f(x),x)-p) \\ &\quad + (1-\alpha)^{-1}\alpha(d(f(x),x)-p)-p) \end{aligned}$$

implies

$$d(f^n(x), f^m(x)) \le 2\alpha^n(1-\alpha)^{-1}(d(f(x),x)-p).$$

Since $2\alpha^n(1-\alpha)^{-1}(d(f(x),x)-p) \to \theta,\ \ n \to \infty$, by Lemma 2.4, $\{f^n(x)\}$ is a Cauchy sequence and so is convergent. Thus, there exists $x^* \in A$, such that

$$\lim_{n\to\infty} f^n(x) = x^*.$$

Now, suppose that $c \gg \theta$ and $\in \gg \theta$. Then there exists

$n_0 \in \mathrm{N}$ such that

$$d(x^*, f^n(x)) \ll c, d(f^n(x), f^m(x)) \ll \in$$

and

$$d(x^*, f^n(x)) \ll \in \text{ for all } n, m \geq n_0. \qquad (2)$$

Now for each $n > n_0$,

$$d(x^*, f(x^*)) - p$$
$$\ll d(x^*, f^n(x)) + d(f^n(x), f(x^*)) - p$$
$$\leq c + d(f(x^*), f^n(x)) - p. \qquad (3)$$

Furthermore, since f is a quasi contraction, we have

$$d(f^n(x), f(x^*)) \leq \alpha(u) \qquad (4)$$

for some

$$u \in \{d(f^{n-1}(x), x^*) - p, d(f^{n-1}(x), f^n(x)) - p,$$
$$d(f^{n-1}(x), f(x^*)) - p, d(x^*, f(x^*)) - p,$$
$$d(f(x^*), f^n(x)) - p\}.$$

If

$$u \in \{d(f^{n-1}(x), x^*) - p, d(f^{n-1}(x), f^n(x)) - p,$$
$$d(f(x^*), f^n(x)) - p\},$$

for infinitely many $n > n_0$, then, by (2), (3) and (4), we get

$$d(x^*, f(x^*)) - p \leq c + \alpha(\in). \qquad (5)$$

Since the inequality (5) is true for each $c \gg \theta$, we get

$$d(x^*, f(x^*)) - p \leq \alpha(\in). \qquad (6)$$

If $u = d(f^{n-1}(x), f(x^*)) - p$,

then

$$d(f^{n-1}(x), f(x^*)) - p$$
$$\leq d(f^{n-1}(x), x^*) + d(x^*, f(x^*)) - p$$

implies

$$\alpha(u) \leq \alpha(d(f^{n-1}(x), x^*))$$
$$+ \alpha(d(x^*, f(x^*)) - p).$$

Now by (2), (3) and (4), we get

$$d(x^*, f(x^*)) - p \leq c + \alpha(\in) + \alpha(d(x^*, f(x^*) - p)$$

and, since $c \gg \theta$ is arbitrary, it follows

$$d(x^*, f(x^*) - p \leq \alpha(\in) + \alpha(d(x^*, f(x^*) - p)$$

i.e. $(1-\alpha)(d(x^*, f(x^*)) - p) \leq \alpha(\in)$. (7)

which implies

$$d(x^*, f(x^*) - p \leq (1-\alpha)^{-1}\alpha(\in). \qquad (8)$$

Finally, if $u = d(x^*, f(x^*)) - p$, then by (3) and (4), we have

$$(1-\alpha)(d(x^*, f(x^*) - p) \leq c. \qquad (9)$$

From (9), we conclude that

$$d(x^*, f(x^*)) - p \leq (1-\alpha)^{-1}(c). \qquad (10)$$

Now, by (6), (8) and (10), for $\in = \dfrac{\in^*}{n}$ and $c = \dfrac{c^*}{n}, n = 1, 2, \ldots,$ we get, respectively,

$$\theta \leq d(x^*, f(x^*)) - p \leq \alpha\left(\frac{\varepsilon^*}{n}\right) = \frac{\alpha(\varepsilon^*)}{n} \to \theta,$$

$n \to \infty$,

$$\theta \leq d(x^*, f(x^*)) - p$$
$$\leq (1-\alpha)^{-1}\left(\frac{c^*}{n}\right)$$
$$= \frac{(1-\alpha)^{-1}(c^*)}{n} \to \theta, \; n \to \infty.$$

and

$$\theta \leq d(x^*, f(x^*)) - p$$
$$\leq (1-\alpha)^{-1}\alpha\left(\frac{\varepsilon^*}{n}\right)$$
$$= \frac{(1-\alpha)^{-1}\alpha(\varepsilon^*)}{n} \to \theta, n \to \infty.$$

Therefore, $d(x^*, f(x^*)) - p = \theta$ gives

$$d(x^*, f(x^*)) = p.$$

If y is another best proximity point of f, i.e.

$$d(y, f(y)) = p,$$

then

$$d(x^*, y) \leq d(x^*, f(x^*)) + d(f(x^*), f(y))$$
$$+ d(f(y), y)$$
$$\leq 2p + \alpha(d(x^*, y) - p)$$
$$= 2p + \alpha(d(x^*, y)) - \alpha p$$
$$\leq 2p + \alpha(d(x^*, y)) - p$$
$$= p + \alpha(d(x^*, y))$$

Hence the result follows.

Corollary 3.3 Let A, B be closed subsets of a complete cone metric space (X, d) with a solid cone P, $int(P) \neq \phi$. Let $f : A \to B$ satisfying

$$d(fx, fy) \leq \alpha d(x, y)$$

with $f(A) \subseteq B$ and $\alpha \in (0,1)$. Then f has unique best proximity point x^* in A.

Taking $A = B = X$ in Theorem 3.1 yields the main result of [10].

Corollary 3.4 Let (X, d) be a complete cone metric space with a solid cone P, $int(P) \neq \phi$. Let $f : X \to X$ be quasi-contraction. Then f has unique fixed point x^* in X

and the iterative sequence $\{f^n x\}$ converges to the fixed point for any $x \in X$.

Taking $A = B = X$ in Corollary 3.3 we obtained the following corollary:

Corollary 3.5 [10] Let (X,d) be a complete cone metric space with a solid cone P, $int(P) \neq \phi$. Let $f : X \to X$ satisfying

$$d(fx, fy) \leq \alpha d(x, y)$$

for some constant $\alpha \in (0,1)$. Then f has a unique fixed point x^* in X, the iterative sequence $\{f^n x\}$ converges to the fixed point.

ACKNOWLEDGEMENT:

The authors are thankful to Dr. Mujahid Abbas for his useful suggestions to improve the quality of the paper.

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