DIRECT PRODUCT OF GENERALIZED CUBIC SETS IN H_v-LA-SEMIGROUPS

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ABSTRACT: This paper aims to explore the structural properties of H_v-LA-semigroups and of the direct product of H_v-LA-semigroups using the idea of generalized cubic sets. It focuses on different types of generalized cubic ideals in H_v-LA-semigroups and in the direct product of H_v-LA-semigroups.

Key Words: H_v-LA-semigroups, Direct product of H_v-LA-semigroups, Cubic sets, Generalized cubic H_v-LA-subsemigroups, Generalized cubic H_v-ideals

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1. INTRODUCTION

F. Marty originated the concept of hyperstructures in 1934, when Marty, [1] characterize hyper gatherings, break down their properties and connected them to assemblies. A lot of papers and several books have been written on hyper structure theory; see [2, 3 and 4]. In 1990, T. Vougiouklis [5] introduced the concept of H_v-structures. After the introduction of H_v-structures, several authors such as Vougiouklis [6, 7 and 8], Spartalis [9, 10, 11 and 12], Spartalis *et a*l. [13], Davvaz [14], Nezhad *et al.* [15] and Hedayati et al. [16, 17] studied different aspects of it. Kazim and Naseeruddin[18], presented the idea of LA-semigroups. Later, Mushtaq [19], and some different mathematicians further explored the structure and added numerous functional effects to the hypothesis of LA-semigroups; see [20, 21, 22, 23, 24, 25 and 26]. Hila et al. [27], initiated the notion of LA-semihypergroups as a generalization of semigroups, semihypergroups, and LA-semigroups. Yaqoob et al. [28] extended the work of Hila and Dine and characterized intra-regular left almost semihypergroups by their hyperideals using pure left identity.

Mushtaq et al. gave the idea of direct product of abel grassmann's groupoids [29]. More recently Gulistan et. al gave the concept of direct products of H_v-LA-semigroups [30]. They proved that if $(H_1,*)$ and $(H_2,\bullet)$ are two H_v-LA-semigroups, then direct product of two H_v-LA-semigroups $(H_1\times H_2,\otimes)$ is again an H_v-LA-semigroup, where $\otimes$ is a hyperoperation on $H_1\times H_2$, defined by

$$(a_1,b_1)\otimes(a_2,b_2)=\{(c,d)\mid c\in a_1*a_2,\ d\in b_1\bullet b_2\},$$

for all $(a_1,b_1),(a_2,b_2)\in H_1\times H_2$. Zadah introduced the concept of fuzzy sets [31]. The fuzzification of hyperstructures was considered by many authors. For instance, Ameri et al. [32 and 33], Fotea et al. [34 and 35], Davvaz et al. [36], Corsini et al. [37]. A. K. Ray introduced the concept of product of fuzzy subgroups in his paper [38]. Aktas et al. introduced the concept of generalized product of fuzzy subgroups and some fundamental properties [39]. Aslam et al. introduced the concept of direct product of fuzzy ideals in LA-semigroup and the direct product of intuitionistic fuzzy set in LA-semigroup and obtained some usful results [40, 41 and 42]. The fuzzification of H_v-structures was also considered by many mathematicians, see Davvaz et al. [43, 44, 45, 46, 47, 48 and 49]. Jun et al. [50] introduced the notion of cubic sub-algebras/ideals in BCK/BCI-algebras, and see also [51, 52, 53 and 54]. See also [55,56,57,58,59,60].

In this paper in section 2, some basic definitions and results of H_v-LA-semigroups have been provided. In section 3, we define the concept of generalized cubic H_v-LA-subsemigroups, generalized cubic H_v-ideals of H_v-LA-semigroups and discuss their basic properties. We show that every $(\in,\in\vee q)$-cubic H_v-LA-subsemigroup is an $(\in,\in\vee q_k)$-cubic H_v-LA-subsemigroup and every $(\in,\in\vee q_k)$-cubic H_v-LA-subsemigroup is an $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$-cubic H_v-LA-subsemigroup, but not conversely. In section 4, defining the concept of cubic ideals and generalized cubic ideals of the direct product of H_v-LA-semigroups we prove some results.

2. SOME BASIC NOTIONS IN H_v-LA-SEMIGROUPS

Throughout the whole article H denotes the H_v-LA-semigroup for simplicity and $\Gamma=(\tilde{\gamma}_1,\gamma_2)$ $\Delta=(\tilde{\delta}_1,\delta_2)$. Following are some basic definitions and results.

Definition 1. *A map* $\circ: S\times S\to \mathbf{P}^*(S)$ *is called a* hyperoperation *or* join operation *on the set* S*, where* S *is a non-empty set and* $\mathbf{P}^*(S)=\mathbf{P}(S)\setminus\{\phi\}$ *denotes the set of all non-empty subsets of* S*. A hypergroupoid is a set* S *together with a (binary) hyperoperation.*

Definition 2 [27, 28]. *A hypergroupoid* $(S,\circ)$ *which is left invertive (non-associative), that is* $(x\circ y)\circ z\,\bar{=}\,(z\circ y)\circ x$, $\forall x,y,z\in S$ *, is called* an *LA-semihypergroup.*

Definition 3[30]. *Let* H *be a non-empty set and* $*$ *be a hyperoperation on* H*. Then* $(H,*)$ *is called an*

H_v-LA-semigroup if it satisfies the weak left invertive law i.e for all $x, y, z \in H$, $(x*y)*z \cap (z*y)*x \neq \phi$.

Example 1[30]. *Consider* $H = \{x, y, z\}$ *and define a hyperoperation* $*$ *on* H *by the following table:*

$*$	x	y	z
x	x	$\{x,z\}$	H
y	$\{x,z\}$	x	x
z	$\{x,y\}$	z	$\{x,z\}$

then $(H,*)$ *is an H_v-LA-semigroup.*

Definition 4[30]. *A non-empty subset* K *of* $(H,*)$ *is said to be an H_v-LA-subsemigroup if it is itself an H_v-LA-semigroup or* $a*b \in K,\ \forall\ a,b \in K$. K *is called proper H_v-LA-subsemigroup if* $K \neq H$.

Definition 5[30]. *A non-empty subset* K *of* $(H,*)$ *is said to be an H_v-ideal of* H *if* $a*K \subseteq K$, *for all* $a \in H$.

Definition 6[30]. *Let* $(H_1,*)$ *and* $(H_2,\bullet)$ *be two H_v-LA-semigroups. Given* $(H_1 \times H_2, \otimes)$, $\otimes$ *is a hyperoperation on* $H_1 \times H_2$, *such that*

$$(a_1,b_1)\otimes(a_2,b_2) = \{(c,d) \mid c \in a_1 * a_2,\ d \in b_1 \bullet b_2\},$$

or all $(a_1,b_1),(a_2,b_2) \in H_1 \times H_2$. *Then we say* $(H_1 \times H_2, \otimes)$ *is the direct product of H_v-LA-semigroups* $(H_1,*)$ *and* $(H_2,\bullet)$.

Proposition 1[30]. *The direct product of two H_v-LA-semigroups is again an H_v-LA-semigroup.*

Proposition 2[30]. *If* $(K,*)$ *and* $(L,\bullet)$ *are two H_v-LA-subsemigroups (ideals) of* $(H_1,*)$ *and* $(H_2,\bullet)$, *respectively, then the direct product* $K \times L$ *is also an H_v-LA-subsemigroup (ideal) of* $(H_1 \times H_2, \otimes)$.

Jun et al. [50] introduced the concept of cubic sets defined on a non-empty set X as objects having the form: $\Im = \{\langle x, \tilde{\eta}_\Im(x), \vartheta_\Im(x)\rangle : x \in X\}$, which is briefly denoted by $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$, where the functions $\tilde{\eta}_\Im : X \to D[0,1]$ and $\vartheta_\Im : X \to [0,1]$.

Definition 7. *Let* $\Im_1 = \langle \tilde{\eta}_{\Im_1}, \vartheta_{\Im_1} \rangle$ *and* $\Im_2 = \langle \tilde{\eta}_{\Im_2}, \vartheta_{\Im_2} \rangle$ *be two cubic sets of* H. *Then*

$$\Im_1 \cap \Im_2 = \left\{\langle x, \tilde{\eta}_{\Im_1\cap\Im_2}(x), \vartheta_{\Im_1\cap\Im_2}(x)\rangle : x \in H\right\}$$
$$= \left\{\langle x, \text{rmin}\{\tilde{\eta}_{\Im_1}(x), \tilde{\eta}_{\Im_2}(x)\}, \max\{\vartheta_{\Im_1}(x), \vartheta_{\Im_2}(x)\}\rangle : x \in H\right\},$$

$$\Im_1 \cup \Im_2 = \left\{\langle x, \tilde{\eta}_{\Im_1\cup\Im_2}(x), \vartheta_{\Im_1\cup\Im_2}(x)\rangle : x \in H\right\}$$
$$= \left\{\langle x, \text{rmax}\{\tilde{\eta}_{\Im_1}(x), \tilde{\eta}_{\Im_2}(x)\}, \min\{\vartheta_{\Im_1}(x), \vartheta_{\Im_2}(x)\}\rangle : x \in H\right\},$$

and

$$\Im_1 * \Im_2 = \left\{\langle x, \tilde{\eta}_{\Im_1*\Im_2}(x), \vartheta_{\Im_1*\Im_2}(x)\rangle : x \in H\right\},$$

where

$$\tilde{\eta}_{\Im_1*\Im_2}(x) = \begin{cases} \text{rsup}_{x\in y\circ z}\{\text{rmin}\{\tilde{\eta}_{\Im_1}(y), \tilde{\eta}_{\Im_2}(z)\}\} & \text{if } x \in y\circ z \\ [0,0] & \text{otherwise} \end{cases}$$

$$\vartheta_{\Im_1*\Im_2}(x) = \begin{cases} \inf_{x\in y\circ z}\{\max\{\vartheta_{\Im_1}(y), \vartheta_{\Im_2}(z)\}\} & \text{if } x \in y\circ z \\ 1 & \text{otherwise.} \end{cases}$$

Denote by $\mathbf{C}(H)$ the family of all cubic sets in H.

Definition 8. Let $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ be a cubic set of H. Then the $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic characteristic function $\mathbf{X}_\Im^{(\Gamma,\Delta)} = \langle \tilde{\eta}_{\mathbf{X}_\Im^{(\Gamma,\Delta)}}, \vartheta_{\mathbf{X}_\Im^{(\Gamma,\Delta)}} \rangle$ of $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ is defined as

$$\tilde{\eta}_{\mathbf{X}_\Im^{(\Gamma,\Delta)}}(x) \succeq \begin{cases} \tilde{\delta}_1 = \tilde{1} & \text{if } x \in \Im \\ \tilde{\gamma}_1 = [0,0] & \text{if } x \notin \Im \end{cases} \quad \text{and} \quad \vartheta_{\mathbf{X}_\Im^{(\Gamma,\Delta)}}(x) \leq \begin{cases} \delta_2 = 0 & \text{if } x \in \Im \\ \gamma_2 = 1 & \text{if } x \notin \Im \end{cases},$$

Where $\tilde{\delta}_1, \tilde{\gamma}_1 \in D(0,1]$ such that $\tilde{\gamma}_1 \prec \tilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

3. GENERALIZED CUBIC IDEALS OF H $_v$-LA-SEMIGROUPS

In this section we define the concept of generalized cubic H_v-LA-subsemigroups, generalized cubic H_v-ideals of H_v-LA-semigroups and discuss some of their basic properties. We show here that every $(\in, \in \vee q)$-cubic H_v-LA-subsemigroup is an $(\in, \in \vee q_K)$-cubic H_v-LA-subsemigroup and every $(\in, \in \vee q_K)$-cubic H_v-LA-subsemigroup is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic H_v-LA-subsemigroup, but not conversely.

Definition 9. *A cubic set* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *of* H *is called* $(\in, \in \vee q)$ *-cubic H_v-LA-subsemigroup if it satisfies*

(S_1) $x_{(\tilde{t}_1, t_2)} \in \Im$ *and* $y_{(\tilde{t}_3, t_4)} \in \Im$ *imply that* $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\}, \max\{t_2,t_4\})} \in \vee q \Im$, *for all* $z \in x*y$.

Definition 10. *A cubic set* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *of* H *is called* $(\in, \in \vee q_K)$ *-cubic H_v-LA-subsemigroup if it satisfies,*

(S_2) $x_{(\tilde{t}_1,t_2)} \in \Im$ *and* $y_{(\tilde{t}_3,t_4)} \in \Im$ *imply that* $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})} \in \vee q_K \Im,$ *for all* $z \in x * y$.

Definition 11. *A cubic set* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *of* H *is called* $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *-LA-subsemigroup if it satisfies,*

(S_3) $x_{(\tilde{t}_1,t_2)} \in_\Gamma \Im$ *and* $y_{(\tilde{t}_3,t_4)} \in_\Gamma \Im$ *imply that* $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})} \in_\Gamma \vee q_\Delta \Im,$ *for all* $z \in x * y$.

Definition 12. *A cubic set* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *of* H *is called* $(\in, \in \vee q)$ *-cubic* H_v *left ideal (resp.,* H_v *right ideal) of* H *if it satisfies,*

(I_1) $y_{(\tilde{t}_1,t_2)} \in \Im$ *and* $x \in H$ *imply that* $z_{(\tilde{t}_1,t_2)} \in \vee q \Im,$ *for all* $z \in x * y$ *(resp.,* $z \in y * x$ *).*

Definition 13. *A cubic set* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *of* H *is called* $(\in, \in \vee q_K)$ *-cubic* H_v *left ideal (resp.,* H_v *right ideal) of* H *if it satisfies,*

(I_2) $y_{(\tilde{t}_1,t_2)} \in \Im$ *and* $x \in H$ *imply that* $z_{(\tilde{t}_1,t_2)} \in \vee q_K \Im,$ *for all* $z \in x * y$ *(resp.,* $z \in y * x$*).*

Definition 14. *A cubic set* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *of* H *is called* $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *left ideal (resp.,* H_v *right ideal) of* H *if it satisfies,*

(I_2) $y_{(\tilde{t}_1,t_2)} \in_\Gamma \Im$ *and* $x \in H$ *imply that* $z_{(\tilde{t}_1,t_2)} \in_\Gamma \vee q_\Delta \Im,$ *for all* $z \in x * y$ *(resp.,* $z \in y * x$*).*

Remark 1. *Every* $(\in, \in \vee q)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in, \in \vee q)$ *-cubic* H_v *ideal) is an* $(\in, \in \vee q_K)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in, \in \vee q_K)$ *-cubic* H_v *ideal) and every* $(\in, \in \vee q_K)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in, \in \vee q_K)$ *-cubic* H_v *ideal) is an* $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *ideal), but not conversely.*

Example 2. *Consider* H_v *-LA-semigroup define in Example 1 and define the cubic sets* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *as*

H	$\tilde{\eta}_\Im$	$\vartheta_\Im$	$\tilde{t}_1 = [0.22, 0.23)$	$\gamma_2 = 0.7$
x	$[0.3, 0.4)$	0.43	$\tilde{t}_2 = [0.24, 0.27)$	$s_1 = 0.6$
y	$[0.41, 0.5)$	0.4	$\tilde{\delta}_1 = \tilde{k}_1 = [0.18, 0.19)$	$s_2 = 0.75$
z	$[0.51, 0.6)$	0.3	$\tilde{\gamma}_1 = [0.15, 0.18)$	$\delta_2 = k_2 = 0.6$

such that $\tilde{\gamma}_1 = [0.15, 0.18) \prec \tilde{\delta}_1 = [0.18, 0.19)$ *and* $\delta_2 = 0.6 < \gamma_2 = 0.7$. *Then*

(i) $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *is an* $(\in_{([0.15,0.18),0.7)}, \in_{([0.15,0.18),0.7)} \vee q_{([0.18,0.19),0.6)})$ *-cubic* H_v *-LA-subsemigroup of* H.

(ii) $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *is not an* $(\in, \in \vee q_{([0.18,0.19),0.6)})$ *-cubic* H_v *-LA-subsemigroup of* H*, as* $\vartheta_\Im(x * y) + 0.75 + 0.6 > 1$ *for every* $x \in H$.

(ii) $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *is not an* $(\in, \in \vee q)$ *-cubic* H_v *-LA-subsemigroup of* H*, as* $\tilde{\eta}_\Im(x * y) + [0.22, 0.23) < \tilde{1}$ *and* $\vartheta_\Im(x * y) + 0.75 > 1$ *for every* $x \in H$. *Similarly the case for generalized cubic ideals can be seen.*

Proposition 3. *For an* H_v *-LA-semigroup, the following hold:*

(i) *Every* $(\in \vee q, \in \vee q)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in \vee q, \in \vee q)$ *-cubic* H_v *ideal) of* H *is an* $(\in, \in \vee q)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in, \in \vee q)$*-cubic* H_v *ideal) of* H.

(ii) *Every* $(\in, \in)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in, \in)$ *-cubic* H_v *ideal) of* H *is an* $(\in, \in \vee q)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in, \in \vee q)$ *-cubic* H_v *ideal) of* H.

(iii) *Every* $(\in \vee q_K, \in \vee q_K)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in \vee q_K, \in \vee q_K)$ *-cubic* H_v *ideal) of* H *is an* $(\in, \in \vee q_K)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in, \in \vee q_K)$ *-cubic* H_v *ideal) of* H.

(iv) *Every* $(\in, \in)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in, \in)$ *-cubic* H_v *ideal) of* H *is an* $(\in, \in \vee q_\kappa)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in, \in \vee q_K)$ *-cubic* H_v *ideal) of* H.

(v) *Every* $(\in_\Gamma \vee q_\Delta, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in_\Gamma \vee q_\Delta, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *ideal) of* H *is an* $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in_\Gamma \vee q_\Delta, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *ideal) of* H.

(vi) *Every* $(\in_\Gamma, \in_\Gamma)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in_\Gamma, \in_\Gamma)$ *-cubic* H_v *ideal) of* H *is an* $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *-LA-subsemigroup (resp.,* $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$ *-cubic* H_v *ideal) of* H.

Proof: (i) Let $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ be an $(\in \vee q, \in \vee q)$ -cubic H $_v$ -LA-subsemigroup of H. Assume that $x_{(\tilde{t}_1,t_2)} \in \Im$ and $y_{(\tilde{t}_3,t_4)} \in \Im,$ where $x, y \in H$ and $\tilde{t}_1, \tilde{t}_3 \in D(0,1]$ and $t_2, t_4 \in [0,1)$. This implies that $x_{(\tilde{t}_1,t_2)} \in \vee q \Im$ and $y_{(\tilde{t}_3,t_4)} \in \vee q \Im.$ This shows that $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})} \in \vee q \Im,$ for all $z \in x * y$ by hypothesis . Hence $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ be an $(\in, \in \vee q)$ -cubic

H $_v$-LA-subsemigroup of H. Similarly the case for $(\in\vee q,\in\vee q)$-cubic H $_v$-ideals of H can be proved ·

(ii) Starightforward.

(iii) Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ be an $(\in\vee q_K,\in\vee q_K)$-cubic H $_v$-LA-subsemigroup of H. Assume that $x_{(\tilde{t}_1,t_2)}\in\Im$ and $y_{(\tilde{t}_3,t_4)}\in\Im$, where $x,y\in H$ and $\tilde{t}_1,\tilde{t}_3\in D(0,1]$ and $t_2,t_4\in[0,1)$. This implies that $x_{(\tilde{t}_1,t_2)}\in\vee q_K\Im$ and $y_{(\tilde{t}_3,t_4)}\in\vee q_K\Im$. This shows that $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in\vee q_K\Im$, for all $z\in x*y$ by hypothesis · Hence $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ be an $(\in,\in\vee q_K)$-cubic H $_v$-LA-subsemigroup of H. Similarly the case for $(\in\vee q,\in\vee q)$-cubic H $_v$-ideals of H can be proved ·

(iv) Starightforward.

(v) Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ be an $(\in_\Gamma\vee q_\Delta,\in_\Gamma\vee q_\Delta)$-cubic H $_v$-LA-subsemigroup of H. Assume that $x_{(\tilde{t}_1,t_2)}\in\Im$ and $y_{(\tilde{t}_3,t_4)}\in\Im$, where $x,y\in H$ and $\tilde{t}_1,\tilde{t}_3\in D(0,1]$ and $t_2,t_4\in[0,1)$. This implies that $x_{(\tilde{t}_1,t_2)}\in_\Gamma\vee q_\Delta\Im$ and $y_{(\tilde{t}_3,t_4)}\in_\Gamma\vee q_\Delta\Im$. This shows that $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in_\Gamma\vee q_\Delta\Im$, for all $z\in x*y$ by hypothesis · Hence $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ is an $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$-cubic H $_v$-LA-subsemigroup of H.

Similarly the case for $(\in_\Gamma\vee q_\Delta,\in_\Gamma\vee q_\Delta)$-cubic H $_v$ -ideals of H can be proved ·

(vi) Starightforward.

Lemma 1. For an H $_v$-LA-semigroup, the following hold:

(i) If A is an H $_v$ -LA-subsemigroup of H (resp., H $_v$ ideal), then cubic characteristic function of $A=\langle\tilde{\eta}_A,\vartheta_A\rangle$ is an $(\in,\in)$-cubic H $_v$-LA-subsemigroup (resp., $(\in,\in)$-cubic H $_v$ ideal) of H.

(ii) If A is an H $_v$-LA-subsemigroup *of H (resp., H $_v$ ideal), then* $(\in,\in\vee q_K)$*-cubic characteristic function* $\mathbf{X}_A=\langle\tilde{\eta}_{\mathbf{X}_A},\vartheta_{\mathbf{X}_A}\rangle$ *of* $A=\langle\tilde{\eta}_A,\vartheta_A\rangle$ *is an* $(\in,\in)$ *-cubic H $_v$-LA-subsemigroup (resp.,* $(\in,\in)$ *-cubic H $_v$ ideal) of* H.

(iii) *If A is an H $_v$ -LA-subsemigroup of H (resp., H $_v$ ideal), then the* $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$*-cubic characteristic function* $\mathbf{X}^{(\Gamma,\Delta)}A=\langle\tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}},\vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}}\rangle$ *of* $A=\langle\tilde{\eta}_A,\vartheta_A\rangle$ *is an* $(\in_\Gamma,\in_\Gamma)$*-cubic H $_v$-LA-subsemigroup (resp.,* $(\in_\Gamma,\in_\Gamma)$*-cubic H $_v$ ideal) of* H.

Proof: (i) Let A be an H $_v$-LA-subsemigroup of H. Assume that $x_{(\tilde{t}_1,t_2)}\in X_A$ and $y_{(\tilde{t}_3,t_4)}\in X_A$, where $x,y\in H$ and $\tilde{t}_1,\tilde{t}_3\in D(0,1]$ and $t_2,t_4\in[0,1)$. This implies that $\tilde{\eta}_{\mathbf{X}_A}(x)\succeq\tilde{t}_1\succ\tilde{0}$, $\vartheta_{\mathbf{X}_A}(x)\le t_2<1$ and $\tilde{\eta}_{\mathbf{X}_A}(y)\succeq\tilde{t}_3\succ\tilde{0}$, $\vartheta_{\mathbf{X}_A}(y)\le t_4<1$. Which then implies that $\tilde{\eta}_{\mathbf{X}_A}(x)=\tilde{\eta}_{\mathbf{X}_A}(y)=\tilde{1}$ and $\vartheta_{\mathbf{X}_A}(x)=\vartheta_{\mathbf{X}_A}(y)=0$. Thus for all $z\in x*y$, we have $z\in X_A$. Which implies that $\tilde{\eta}_{\mathbf{X}_A}(z)=\tilde{1}\succeq\text{rmin}\{\tilde{t}_1,\tilde{t}_3\}$ and $\vartheta_{\mathbf{X}_A}(z)=0\le\max\{t_2.t_4\}$. Thus $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in\Im$, for all $z\in x*y$. Hence cubic characteristic function $X_A=\langle\tilde{\eta}_{\mathbf{X}_A},\vartheta_{\mathbf{X}_A}\rangle$ of $A=\langle\tilde{\eta}_A,\vartheta_A\rangle$ is an $(\in,\in)$-cubic H $_v$-LA-subsemigroup of H. Similarly the case for $(\in,\in)$-cubic H $_v$-ideals of H can be proved.

(ii) Let A be an H $_v$-LA-subsemigroup of H. Assume that $x_{(\tilde{t}_1,t_2)}\in X_A$ and $y_{(\tilde{t}_3,t_4)}\in X_A$, where $x,y\in H$ and $\tilde{t}_1,\tilde{t}_3\in D(0,1]$ and $t_2,t_4\in[0,1)$. $\tilde{\eta}_{\mathbf{X}_A}(x)\succeq\tilde{t}_1\succ\tilde{0}$, $\vartheta_{\mathbf{X}_A}(x)\le t_2<1$ and $\tilde{\eta}_{\mathbf{X}_A}(y)\succeq\tilde{t}_3\succ\tilde{0}$, $\vartheta_{\mathbf{X}_A}(y)\le t_4<1$. Which then implies that $\tilde{\eta}_{\mathbf{X}_A}(x)=\tilde{\eta}_{\mathbf{X}_A}(y)=\frac{\tilde{1}-\tilde{k}_1}{2}$ and $\vartheta_{\mathbf{X}_A}(x)=\vartheta_{\mathbf{X}_A}(y)=0$. Thus for all $z\in x*y$, we have $z\in X_A$. Which implies that $\tilde{\eta}_{\mathbf{X}_A}(z)=\frac{\tilde{1}-\tilde{k}_1}{2}\succeq\text{rmin}\{\tilde{t}_1,\tilde{t}_3\}$ and $\vartheta_{\mathbf{X}_A}(x)=\frac{1-k_2}{2}\le\max\{t_2,t_4\}$. Thus $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in\Im$, for all $z\in x*y$. Hence the $(\in,\in\vee q_K)$-cubic characteristic function $X_A=\langle\tilde{\eta}_{\mathbf{X}_A},\vartheta_{\mathbf{X}_A}\rangle$ of $A=\langle\tilde{\eta}_A,\vartheta_A\rangle$ is an $(\in,\in)$-cubic H $_v$-LA-subsemigroup of H. Similarly the case for $(\in,\in)$-cubic H $_v$-ideals of H can be proved.

(iii) Let A be an H $_v$-LA-subsemigroup of H. Assume that $x_{(\tilde{t}_1,t_2)}\in_\Gamma\mathbf{X}^{(\Gamma,\Delta)}A$ and $y_{(\tilde{t}_3,t_4)}\in_\Gamma\mathbf{X}^{(\Gamma,\Delta)}A$, where $x,y\in H$ and $\tilde{t}_1,\tilde{t}_3\in D(0,1]$ and $t_2,t_4\in[0,1)$. This implies that $\tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}}(x)\succeq\tilde{t}_1\succ\tilde{\gamma}_1=\tilde{0}$, $\vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}}(x)\le t_2<\gamma_2=1$ and $\tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}}(x)\succeq\tilde{t}_1\succ\tilde{\gamma}_3=\tilde{0}$, $\vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}}(y)\le t_4<\gamma_2=1$. Which then implies that $\tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}}(x)=\tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}}(y)=\tilde{\delta}_1=(1,1]$ and $\vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}}(x)=\vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}}(y)=\delta_2=0$. Thus for all $z\in x*y$, we have $z \in_\Gamma \mathbf{X}^{(\Gamma,\Delta)} A$. Which implies that $\tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}}(z) = \tilde{\delta}_1 = (1,1] \succeq \text{rmin}\{\tilde{t}_1, \tilde{t}_3\}$ and $\vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}}(x) = \delta_2 = 0 \le \max\{t_2, t_4\}$. Thus $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})} \in_\Gamma \Im$, for all $z \in x*y$. Hence $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic characteristic function $\mathbf{X}_A^{(\Gamma,\Delta)} = \left\langle \tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}}, \vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}} \right\rangle$ of $A = \langle \tilde{\eta}_A, \vartheta_A \rangle$ is an $(\in_\Gamma, \in_\Gamma)$-cubic H $_v$-LA-subsemigroup of H. Similarly the case for $(\in_\Gamma, \in_\Gamma)$-cubic H $_v$-ideals of H can be proved.

Lemma 2. *For an H $_v$-LA-semigroup, the following hold:*

(i) *Cubic characteristic function $\mathbf{X}_A$ of A is an $(\in, \in \vee q)$-cubic H $_v$-LA-subsemigroup (resp., $(\in,\in)$-cubic H $_v$ ideal) if and only if A is an H $_v$-LA-subsemigroup (resp., H $_v$ ideal) of H.*

(ii) *$(\in, \in \vee q_\kappa)$-cubic characteristic function of $\mathbf{X}_A$ of A is an $(\in, \in \vee q_\kappa)$-cubic H $_v$-LA-subsemigroup(resp., $(\in, \in \vee q_\kappa)$-cubic H $_v$ ideal) if and only if A is an H $_v$-LA-subsemigroup (resp., H $_v$ ideal) of H.*

(iii) *$(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic characteristic function $\mathbf{X}_A^{(\Gamma,\Delta)}$ of A is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic H $_v$-LA-subsemigroup (resp., $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic H $_v$ ideal) if and only if A is an H $_v$-LA-subsemigroup (resp., H $_v$ ideal) of H.*

Proof: (i) Let us assume that cubic characteristic function $\mathbf{X}_A = \left\langle \tilde{\eta}_{\mathbf{X}_A}, \vartheta_{\mathbf{X}_A} \right\rangle$ of $A = \langle \tilde{\eta}_A, \vartheta_A \rangle$ is an $(\in, \in \vee q)$-cubic H $_v$-LA-subsemigroup. Assume that $x, y \in A$ then $x_{(\tilde{1},0)} \in \mathbf{X}_A$ and $y_{(\tilde{1},0)} \in \mathbf{X}_A$. This implies that $z_{(\text{rmin}\{\tilde{1},\tilde{1}\},\max\{0,0\})} \in \vee q \mathbf{X}_A$, for all $z \in x*y$. Which implies that $\tilde{\eta}_{\mathbf{X}_A}(z) = \tilde{1}$, $\vartheta_{\mathbf{X}_A}(z) = 0$. So $z \in x*y \in A$. Thus A is an H $_v$-LA-subsemigroup of H. Conversely let A is an H $_v$-LA-subsemigroup of H, then by Lemma 1, $\mathbf{X}_A = \left\langle \tilde{\eta}_{\mathbf{X}_A}, \vartheta_{\mathbf{X}_A} \right\rangle$ of $A = \langle \tilde{\eta}_A, \vartheta_A \rangle$ is an $(\in,\in)$-cubic H $_v$-LA-subsemigroup. Then by Proposition 3, $\mathbf{X}_A = \left\langle \tilde{\eta}_{\mathbf{X}_A}, \vartheta_{\mathbf{X}_A} \right\rangle$ of $A = \langle \tilde{\eta}_A, \vartheta_A \rangle$ is an $(\in, \in \vee q)$-cubic H $_v$-LA-subsemigroup of H. Similarly the case for $(\in,\in)$-cubic H $_v$-ideals of H can be proved.

(ii) Let us assume that $(\in, \in \vee q_{q_K})$-cubic characteristic function $\mathbf{X}_A = \left\langle \tilde{\eta}_{\mathbf{X}_A}, \vartheta_{\mathbf{X}_A} \right\rangle$ of $A = \langle \tilde{\eta}_A, \vartheta_A \rangle$ is an $(\in, \in \vee q_k)$-cubic H $_v$-LA-subsemigroup. Assume that $x, y \in A$ then $x_{\left(\frac{\tilde{1}-\tilde{k}_1}{2}, \frac{1-k_2}{2}\right)} \in \mathbf{X}_A$ and $y_{\left(\frac{\tilde{1}-\tilde{k}_1}{2}, \frac{1-k_2}{2}\right)} \in \mathbf{X}_A$. This implies that $z_{\left(\text{rmin}\{\frac{\tilde{1}-\tilde{k}_1}{2}, \frac{\tilde{1}-\tilde{k}_1}{2}\}, \max\{\frac{1-k_2}{2}, \frac{1-k_2}{2}\}\right)} \in \vee q_{q_K} \mathbf{X}_A$, for all $z \in x*y$. Which implies that $\tilde{\eta}_{\mathbf{X}_A}(z) = \frac{\tilde{1}-\tilde{k}_1}{2}$, $\vartheta_{\mathbf{X}_A}(z) = \frac{1-k_2}{2}$. So $z \in x*y \in A$. Thus A is an H $_v$-LA-subsemigroup of H. Conversely let A is an H $_v$-LA-subsemigroup of H, then by Lemma 1, $\mathbf{X}_A = \left\langle \tilde{\eta}_{\mathbf{X}_A}, \vartheta_{\mathbf{X}_A} \right\rangle$ of $A = \langle \tilde{\eta}_A, \vartheta_A \rangle$ is an $(\in,\in)$-cubic H $_v$-LA-subsemigroup. Then by Proposition 3, $\mathbf{X}_A = \left\langle \tilde{\eta}_{\mathbf{X}_A}, \vartheta_{\mathbf{X}_A} \right\rangle$ of $A = \langle \tilde{\eta}_A, \vartheta_A \rangle$ is an $(\in, \in \vee q_k)$-cubic H $_v$-LA-subsemigroup of H. Similarly the case for $(\in,\in)$-cubic H $_v$-ideals of H can be proved.

(iii) Let us assume that cubic characteristic function $\mathbf{X}_A^{(\Gamma,\Delta)} = \left\langle \tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}}, \vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}} \right\rangle$ of $A = \langle \tilde{\eta}_A, \vartheta_A \rangle$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic H $_v$-LA-subsemigroup. Assume that $x, y \in A$ then $x_{((1,1],0)} \in_\Gamma \mathbf{X}_A^{(\Gamma,\Delta)}$ and $y_{((1,1],0)} \in_\Gamma \mathbf{X}_A^{(\Gamma,\Delta)}$. This implies that $z_{(\text{rmin}\{(1,1],(1,1]\},\max\{0,0\})} = z_{((1,1],0)} \in_\Gamma \vee q_\Delta \mathbf{X}_A^{(\Gamma,\Delta)}$, for all $z \in x*y$. Which implies that $\tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}}(z) \succeq (1,1] = \tilde{\delta}_1$, $\vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}}(z) \le 0 = \delta_2$. So $z \in x*y \in A$. Thus A is an H $_v$-LA-subsemigroup of H. Conversely let A is an H $_v$-LA-subsemigroup of H, then by Lemma 1, $\mathbf{X}_A^{(\Gamma,\Delta)}$ of A is an $(\in_\Gamma, \in_\Gamma)$-cubic H $_v$-LA-subsemigroup. Then by Proposition 3, $\mathbf{X}_A^{(\Gamma,\Delta)} = \left\langle \tilde{\eta}_{\mathbf{X}_A^{(\Gamma,\Delta)}}, \vartheta_{\mathbf{X}_A^{(\Gamma,\Delta)}} \right\rangle$ of $A = \langle \tilde{\eta}_A, \vartheta_A \rangle$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic H $_v$-LA-subsemigroup of H. Similarly the case for $(\in,\in)$-cubic H $_v$-ideals of H can be proved.

Theorem 1. *For an H $_v$-LA-semigroup, the following hold:*

(i) *A cubic set $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ of H is an $(\in, \in \vee q)$-cubic H $_v$-ideal of H if and only if*

(a)

$$\left\{\underset{z \in x*y}{\text{rinf}}\, \tilde{\eta}_\Im(z)\right\} \succeq \text{rmin}\{\text{rmax}\{\tilde{\eta}_\Im(x), \tilde{\eta}_\Im(y)\}, (0.5, 0.5]\},$$

(b)

$$\left\{\sup_{z \in x*y} \vartheta_\Im(z)\right\} \le \max\left\{\min\left\{\vartheta_\Im(x), \vartheta_\Im(y)\right\}, 0.5\right\}.$$

(ii) *A cubic set $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ of H is an $(\in, \in \vee q_\kappa)$-cubic H $_v$-ideal of H if and only if*

(a)

$$\left\{\operatorname{rinf}_{z\in x*y}\tilde{\eta}_{\Im}(z)\right\}\succeq \operatorname{rmin}\{\operatorname{rmax}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y)\},\tfrac{\tilde{1}-\tilde{k}_1}{2}\},$$

(b)

$$\left\{\sup_{z\in x*y}\vartheta_{\Im}(z)\right\}\leq \max\left\{\min\left\{\vartheta_{\Im}(x),\vartheta_{\Im}(y)\right\},\tfrac{1-k_2}{2}\right\}.$$

(iii) *A cubic set* $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ *of* H *is an is an* $(\in_{\Gamma},\in_{\Gamma}\vee q_{\Delta})$*-cubic H* $_v$*-ideal of* H *if and only if*

(a)

$$\operatorname{rmax}\left\{\operatorname{rinf}_{z\in x*y}\tilde{\eta}_{\Im}(z),\tilde{\gamma}_1\right\}\succeq \operatorname{rmin}\{\operatorname{rmax}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y)\},\tilde{\delta}_1\},$$

(b)

$$\min\left\{\sup_{z\in x*y}\vartheta_{\Im}(z),\gamma_2\right\}\leq \max\left\{\min\left\{\vartheta_{\Im}(x),\vartheta_{\Im}(y)\right\},\delta_2\right\},$$

where $\tilde{\delta}_1,\tilde{\gamma}_1\in D(0,1]$ *such that* $\tilde{\gamma}_1\prec\tilde{\delta}_1$, *and* $\delta_2,\gamma_2\in[0,1)$ *such that* $\delta_2<\gamma_2$.

Proof: (i) Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ is an $(\in,\in\vee q)$-cubic H $_v$-ideal of H. Let there exist $x,y\in H$, such that (a) and (b) are not valid. So for some $\tilde{p}\in D(0,1]$ such that $q\in[0,1)$, we have

$$\left\{\operatorname{rinf}_{z\in x*y}\tilde{\eta}_{\Im}(z)\right\}\prec\tilde{p}\preceq\operatorname{rmin}\{\operatorname{rmax}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y)\},(0.5,0.5]\},$$

$$\left\{\sup_{z\in x*y}\vartheta_{\Im}(z)\right\}\geq q>\max\left\{\min\left\{\vartheta_{\Im}(x),\vartheta_{\Im}(y)\right\},0.5\right\}.$$

Now if

$$\operatorname{rmax}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y)\}\prec(0.5,0.5]\text{ and }\min\{\vartheta_{\Im}(x),\vartheta_{\Im}(y)\}>0.5,$$

then we have

$$\left\{\operatorname{rinf}_{z\in x*y}\tilde{\eta}_{\Im}(z)\right\}\prec\tilde{p}\preceq\operatorname{rmax}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y)\},$$

$$\left\{\sup_{z\in x*y}\vartheta_{\Im}(z)\right\}\geq q>\min\{\vartheta_{\Im}(x),\vartheta_{\Im}(y)\}.$$

Then $x_{(\tilde{p},q)}\in\Im$ and $y_{(\tilde{p},q)}\in\Im$, but $z_{(\tilde{p},q)}\overline{\in}\Im$. Also

$$\left\{\operatorname{rinf}_{z\in x*y}\tilde{\eta}_{\Im}(z)\right\}+\tilde{p}\prec(0.5,0.5]+(0.5,0.5]=(1,1],\left\{\sup_{z\in x*y}\vartheta_{\Im}(z)\right\}+q>0.5+0.5=1.$$

So $z_{(\tilde{p},q)}\overline{q}\Im$. Hence $z_{(\tilde{p},q)}\overline{\in\vee q}\Im$, which is contradiction.

On the other side if

$$\operatorname{rmax}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y)\}\succeq(0.5,0.5],\ \min\{\vartheta_{\Im}(x),\vartheta_{\Im}(y)\}\leq 0.5,$$

then we have

$$\left\{\operatorname{rinf}_{z\in x*y}\tilde{\eta}_{\Im}(z)\right\}\prec(0.5,0.5],\left\{\sup_{z\in x*y}\vartheta_{\Im}(z)\right\}>0.5.$$

Then $x_{((0.5,0.5],0.5)}\in\Im$ and $y_{((0.5,0.5],0.5)}\in\Im$, but $z_{((0.5,0.5],0.5)}\overline{\in}\Im$. Also

$$\left\{\operatorname{rinf}_{z\in x*y}\tilde{\eta}_{\Im}(z)\right\}+(0.5,0.5]\prec(0.5,0.5]+(0.5,0.5]=(1,1],\left\{\sup_{z\in x*y}\vartheta_{\Im}(z)\right\}+0.5>0.5+0.5=1.$$

So $z_{((0.5,0.5],0.5)}\overline{q}\Im$. Hence $z_{((0.5,0.5],0.5)}\overline{\in\vee q}\Im$, which is again a contradiction. Hence (a) and (b) are valid.

Conversely, suppose that (a) and (b) are valid. Let $x_{(\tilde{t}_1,t_2)}\in\Im$ and $y_{(\tilde{t}_3,t_4)}\in\Im$, where $\tilde{t}_1,\tilde{t}_3\in D(0,1]$ and $t_2,t_4\in[0,1)$. This implies that

$$\tilde{\eta}_{\Im}(x)\succeq\tilde{t}_1,\vartheta_{\Im}(x)\leq t_2\text{ and }\tilde{\eta}_{\Im}(y)\succeq\tilde{t}_3,\vartheta_{\Im}(y)\leq t_4.$$

Now by hypothesis we have

$$\left\{\operatorname{rinf}_{z\in x*y}\tilde{\eta}_{\Im}(z)\right\}\succeq\operatorname{rmin}\{\operatorname{rmax}\{\tilde{t}_1,\tilde{t}_3\},(0.5,0.5]\},\left\{\sup_{z\in x*y}\vartheta_{\Im}(z)\right\}\leq\max\left\{\min\{t_2,t_4\},0.5\right\}.$$

If

$$\operatorname{rmax}\{\tilde{t}_1,\tilde{t}_3\}\preceq(0.5,0.5]\text{ and }\min\{t_2,t_4\}\geq 0.5,$$

then $x_{(\operatorname{rmin}\{\tilde{t}_1,t_2\},\ \max\{t_2,t_4\})}\in\Im$. On the other hand if

$$\operatorname{rmax}\{\tilde{t}_1,\tilde{t}_3\}\succ(0.5,0.5]\text{ and }\min\{t_2,t_4\}<0.5,$$

then $x_{(\operatorname{rmin}\{\tilde{t}_1,t_2\},\ \max\{t_2,t_4\})}q\Im$. Hence

$x_{(\operatorname{rmin}\{\tilde{t}_1,t_2\},\ \max\{t_2,t_4\})}\in\vee q\Im$. Thus $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ is an $(\in,\in\vee q)$-cubic H $_v$-ideal of H.

(ii) Starightforward.

(iii) Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ is an $(\in_{\Gamma},\in_{\Gamma}\vee q_{\Delta})$-cubic H $_v$-ideal. Let there exist $x,y\in H$, such that (a) and (b) are not valid. Then there exist $z\in x*y$ such that

$$\operatorname{rmax}\{\tilde{\eta}_{\Im}(z),\tilde{\gamma}_1\}\prec\operatorname{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y),\tilde{\delta}_1\},\min\{\vartheta_{\Im}(z),\gamma_2\}>t_2\max\{\vartheta_{\Im}(x),\vartheta_{\Im}(y),\delta_2\}.$$

Choose $\tilde{t}_1,\tilde{\delta}_1\in D(0,1],\ t_2,\delta_2\in[0,1)$ such that

$$\operatorname{rmax}\left\{\inf_{z\in x*y}\tilde{\eta}_{\Im}(z),\tilde{\gamma}_1\right\}\prec\tilde{t}_1\operatorname{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y),\tilde{\delta}_1\},\min\left\{\sup_{z\in x*y}\vartheta_{\Im}(z),\gamma_2\right\}>t_2\geq\max\{\vartheta_{\Im}(x),\vartheta_{\Im}(y),\delta_2\}.$$

Then

$$\operatorname{rmax}\left\{r\inf_{z\in x*y}\tilde{\eta}_{\Im}(z),\tilde{\gamma}_1\right\}\prec\tilde{t}_1\Rightarrow r\inf_{z\in x*y}\tilde{\eta}_{\Im}(z)\prec\tilde{t}_1\prec\tilde{\gamma}_1,\min\left\{\sup_{z\in x*y}\vartheta_{\Im}(z),\gamma_2\right\}>t_2\Rightarrow\sup_{z\in x*y}\vartheta_{\Im}(z)>t_2>\gamma_2,$$

then $(z)_{\langle\tilde{t}_1,\,t_2\rangle}\overline{\in_{(\tilde{\gamma}_1,\gamma_1)}\vee q_{(\tilde{\delta}_1,\delta_2)}}\Im$ for $z\in x*y$. On the other hand if

$$\tilde{t}_1\preceq\operatorname{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y),\tilde{\delta}_1\},\ t_2\geq\max\{\vartheta_{\Im}(x),\vartheta_{\Im}(y),\delta_2\}$$

we get

$$\tilde{\eta}_{\Im}(x)\succeq\tilde{t}_1\succ\tilde{\gamma}_1,\tilde{\eta}_{\Im}(y)\succeq\tilde{t}_1\succ\tilde{\gamma}_1,\ \vartheta_{\Im}(x)\leq t_2<\gamma_2,\vartheta_{\Im}(y)\leq t_2<\gamma_2,$$

then $x_{\langle\tilde{t}_1,\,t_2\rangle}\in_{(\tilde{\gamma}_1,\gamma_1)}\Im$ and $y_{\langle\tilde{t}_1,\,t_2\rangle}\in_{(\tilde{\gamma}_1,\gamma_1)}\Im$ but $z_{\langle\tilde{t}_1\ t_2\rangle}\overline{\in_{(\tilde{\gamma}_1,\gamma_1)}\vee q}_{(\tilde{\delta}_1,\delta_2)}\Im$ for $z\in x*y$. Which is contradiction to the hypothesis. Hence (a) and (b) are valid. Conversely assume that there exists $x\in S$, and

$\tilde{t}_1, \tilde{\delta}_1 \in D(0,1]$, $t_2, \delta_2 \in [0,1)$ such that $x_{(\tilde{t},s)} \in_{(\tilde{\gamma}_1,\gamma_1)} \Im$, and $y_{(\tilde{t}_1,s_1)} \in_{(\tilde{\gamma}_1,\gamma_1)} \Im$. This implies that

$$\tilde{\eta}_\Im(x) \succeq \tilde{t} \succ \tilde{\gamma}_1, \vartheta_\Im(x) \le s < \gamma_1,\ \tilde{\eta}_\Im(y) \succeq \tilde{t}_1 \succ \tilde{\gamma}_1, \vartheta_\Im(y) \le s_1 < \gamma_1.$$

So

$$\text{rmax}\left\{r\inf_{z\in x*y} \tilde{\eta}_\Im(z), \tilde{\gamma}_1\right\} \succeq \text{rmin}\{\tilde{\eta}_\Im(x), \tilde{\eta}_\Im(y), \tilde{\delta}_1\} \succeq \text{rmin}\{\tilde{t}, \tilde{t}_1, \tilde{\delta}_1\}$$

$$\min\left\{\sup_{z\in x*y} \vartheta_\Im(z), \gamma_2\right\} \le \max\{\vartheta_\Im(x), \vartheta_\Im(y), \delta_2\} \le \max\{s, s_1, \delta_2\}.$$

We have the following two cases.

(i) If $\{\tilde{t}, \tilde{t}\} \le \tilde{\delta}_1$ and $\{s, s_1\} \ge \delta_2$, then

$$\text{rinf}_{z\in x*y} \tilde{\eta}_\Im(z) \succeq \text{rmin}\{\tilde{t}, \tilde{t}_1\} \succ \tilde{\gamma}_1,\ \sup_{z\in x*y} \vartheta_\Im(z) \le \max\{s, s_1\} < \gamma_2.$$

This implies that $z_{(\text{rmin}\{\tilde{t},\tilde{t}_1\}, \max\{s,s_1\})} \in_{(\tilde{\gamma}_1,\gamma_1)} \Im$.

(ii) If $\{\tilde{t}, \tilde{t}_1\} > \tilde{\delta}_1$ and $\{s, s_1\} < \delta_2$, then

$$\text{rinf}_{z\in x*y} \tilde{\eta}_\Im(z) + \text{rmax}\{\tilde{t}, \tilde{t}_1\} \succ 2\tilde{\delta}_1,\ \sup_{z\in x*y} \vartheta_\Im(z) + \min\{s, s_1\} < 2\delta_2.$$

This implies that $z_{(\text{rmin}\{\tilde{t},\tilde{t}_1\}, \max\{\{s,s_1\}\})} q_{(\tilde{\delta}_1,\delta_2)} \Im$. Hence from both the cases we get

$z_{(\min\{\tilde{t},\tilde{t}_1\}, \max\{\{s,s_1\}\})} \in_{(\tilde{\gamma}_1,\gamma_1)} \vee q_{(\tilde{\delta}_1,\delta_2)} \Im$. Thus $\Im = (\tilde{\eta}_\Im, \vartheta_\Im)$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic H $_v$-ideal of S.

Theorem 2. *For an* H_v*-LA-semigroup, the following hold:*

(i) *A cubic set* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *of* H *is an* $(\in, \in \vee q)$*-cubic* H_v*-ideal of* H *if and only if the non-empty cubic level set* $U(\Im; \tilde{t}, s)$ *is an* H_v*-ideal of* H.

(ii) *A cubic set* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *of* H *is an* $(\in, \in \vee q_\kappa)$*-cubic* H_v*-ideal of* H *if and only if the non-empty cubic level set* $U(\Im; \tilde{t}, s)$ *is an* H_v*-ideal of* H *for* $\tilde{t} \in D(0, \frac{\tilde{1}-\tilde{k}_1}{2}]$ *and* $s \in [\frac{1-k_2}{2}, 1)$.

(iii) *A cubic set* $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ *of* H *is an* $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$*-cubic* H_v*-ideal of* H *if and only if the non-empty cubic level set* $U(\Im; (\tilde{t}, \tilde{\gamma}_1), (s, \gamma_2))$ *is an* H_v*-ideal of* H *for* $\tilde{t} \in D(0,1]$ *and* $s \in [0,1)$.

Proof: (i) Let $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ is an $(\in, \in \vee q)$-cubic H $_v$-ideal of H. Let there exist $x, y \in H$, such that (a) and (b) are not valid. So for some $\tilde{p} \in D(0,1]$ such that $q \in [0,1)$, we have

$$\left\{\text{rinf}_{z\in x*y} \tilde{\eta}_\Im(z)\right\} \prec \tilde{p} \preceq \text{rmin}\{\text{rmax}\{\tilde{\eta}_\Im(x), \tilde{\eta}_\Im(y)\}, (0.5, 0.5]\},$$

$$\left\{\sup_{z\in x*y} \vartheta_\Im(z)\right\} \ge q > \max\{\min\{\vartheta_\Im(x), \vartheta_\Im(y)\}, 0.5\}.$$

Now if

$$\text{rmax}\{\tilde{\eta}_\Im(x), \tilde{\eta}_\Im(y)\} \prec (0.5, 0.5] \text{ and } \min\{\vartheta_\Im(x), \vartheta_\Im(y)\} > 0.5,$$

then we have

$$\left\{\text{rinf}_{z\in x*y} \tilde{\eta}_\Im(z)\right\} \prec \tilde{p} \preceq \text{rmax}\{\tilde{\eta}_\Im(x), \tilde{\eta}_\Im(y)\}, \left\{\sup_{z\in x*y} \vartheta_\Im(z)\right\} \ge q > \min\{\vartheta_\Im(x), \vartheta_\Im(y)\}.$$

Then $x_{(\tilde{p},q)} \in \Im$ and $y_{(\tilde{p},q)} \in \Im$, but $z_{(\tilde{p},q)} \overline{\in} \Im$. Also

$$\left\{\text{rinf}_{z\in x*y} \tilde{\eta}_\Im(z)\right\} + \tilde{p} \prec (0.5, 0.5] + (0.5, 0.5] = (1,1], \left\{\sup_{z\in x*y} \vartheta_\Im(z)\right\} + q > 0.5 + 0.5 = 1.$$

So $z_{(\tilde{p},q)} \overline{q} \Im$.

Hence $z_{(\tilde{p},q)} \overline{\in \vee q} \Im$, which is a contradiction. On the other side if

$$\text{rmax}\{\tilde{\eta}_\Im(x), \tilde{\eta}_\Im(y)\} \succeq (0.5, 0.5],\ \min\{\vartheta_\Im(x), \vartheta_\Im(y)\} \le 0.5,$$

then we have

$$\left\{\text{rinf}_{z\in x*y} \tilde{\eta}_\Im(z)\right\} \prec (0.5, 0.5],\ \left\{\sup_{z\in x*y} \vartheta_\Im(z)\right\} > 0.5.$$

Then $x_{((0.5,0.5],0.5)} \in \Im$ and $y_{((0.5,0.5],0.5)} \in \Im$, but $z_{((0.5,0.5],0.5)} \overline{\in} \Im$. Also

$$\left\{\text{rinf}_{z\in x*y} \tilde{\eta}_\Im(z)\right\} + (0.5, 0.5] \prec (0.5, 0.5] + (0.5, 0.5] = (1,1], \left\{\sup_{z\in x*y} \vartheta_\Im(z)\right\} + 0.5 > 0.5 + 0.5 = 1.$$

So $z_{((0.5,0.5],0.5)} \overline{q} \Im$. Hence $z_{((0.5,0.5],0.5)} \overline{\in \vee q} \Im$, which is again contradiction. Hence (a) and (b) are valid. Conversely, suppose that (a) and (b) are valid. Let $x_{(\tilde{t}_1,t_2)} \in \Im$ and $y_{(\tilde{t}_3,t_4)} \in \Im$, where $\tilde{t}_1, \tilde{t}_3 \in D(0,1]$ and $t_2, t_4 \in [0,1)$. This implies that

$$\tilde{\eta}_\Im(x) \succeq \tilde{t}_1, \vartheta_\Im(x) \le t_2 \text{ and } \tilde{\eta}_\Im(y) \succeq \tilde{t}_3, \vartheta_\Im(y) \le t_4.$$

Now by hypothesis we have

$$\left\{\text{rinf}_{z\in x*y} \tilde{\eta}_\Im(z)\right\} \succeq \text{rmin}\{\text{rmax}\{\tilde{t}_1, \tilde{t}_3\}, (0.5, 0.5]\}, \left\{\sup_{z\in x*y} \vartheta_\Im(z)\right\} \le \max\{\min\{t_2, t_4\}, 0.5\}.$$

If

$$\text{rmax}\{\tilde{t}_1, \tilde{t}_3\} \preceq (0.5, 0.5] \text{ and } \min\{t_2, t_4\} \ge 0.5,$$

then $x_{(\text{rmin}\{\tilde{t}_1,t_2\},\ \max\{t_2,t_4\})} \in \Im$. On the other hand if

$$\text{rmax}\{\tilde{t}_1, \tilde{t}_3\} \succ (0.5, 0.5] \text{ and } \min\{t_2, t_4\} < 0.5,$$

then $x_{(\text{rmin}\{\tilde{t}_1,t_2\},\ \max\{t_2,t_4\})} q \Im$. Hence $x_{(\text{rmin}\{\tilde{t}_1,t_2\},\ \max\{t_2,t_4\})} \in \vee q \Im$. Thus $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ is an $(\in, \in \vee q)$-cubic H $_v$-ideal of H.

(ii) Straightforward.

(iii) Let $\Im = \langle \tilde{\eta}_\Im, \vartheta_\Im \rangle$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic H $_v$-ideal. Let there exist $x, y \in H$, such that (a) and (b) are not valid. Then there exist $z \in x * y$ such that

$$\text{rmax}\{\tilde{\eta}_\Im(z), \tilde{\gamma}_1\} \prec \text{rmin}\{\tilde{\eta}_\Im(x), v_\Im(y), \tilde{\delta}_1\}, \min\{\vartheta_\Im(z), \gamma_2\} > t_2 \max\{\vartheta_\Im(x), \vartheta_\Im(y), \delta_2\}.$$

Choose $\tilde{t}_1, \tilde{\delta}_1 \in D(0,1]$, $t_2, \delta_2 \in [0,1)$ such that

$\text{rmax}\left\{\inf_{z\in x*y}\tilde{\eta}_{\Im}(z),\tilde{\gamma}_1\right\}\prec\tilde{t}_1\preceq\text{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y),\tilde{\delta}_1\},\min\left\{\sup_{z\in x*y}\vartheta_{\Im}(z),\gamma_2\right\}>t_2\geq\max\{\vartheta_{\Im}(x),\vartheta_{\Im}(y),\delta_2\}.$

Then

$\text{rmax}\left\{\inf_{z\in x*y}\tilde{\eta}_{\Im}(z),\tilde{\gamma}_1\right\}\prec\tilde{t}_1\Rightarrow\inf_{z\in x*y}\tilde{\eta}_{\Im}(z)<\tilde{t}_1<\tilde{\gamma}_1,\min\left\{\sup_{z\in x*y}\vartheta_{\Im}(z),\gamma_2\right\}>t_2\Rightarrow\sup_{z\in x*y}\vartheta_{\Im}(z)>t_2>\gamma_2,$

then $(z)_{\langle\tilde{t}_1,t_2\rangle}\overline{\in_{(\tilde{\gamma}_1,\gamma_1)}\vee q_{(\tilde{\delta}_1,\delta_2)}}\Im$ for $z\in x*y$. On the other hand if

$\tilde{t}_1\preceq\text{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y),\tilde{\delta}_1\},\ t_2\geq\max\{\vartheta_{\Im}(x),\vartheta_{\Im}(y),\delta_2\}$

we get

$\tilde{\eta}_{\Im}(x)\succeq\tilde{t}_1>\tilde{\gamma}_1,\tilde{\eta}_{\Im}(y)\succeq\tilde{t}_1>\tilde{\gamma}_1,\ \vartheta_{\Im}(x)\leq t_2<\gamma_2,\vartheta_{\Im}(y)\leq t_2<\gamma_2,$

then $x_{\langle\tilde{t}_1,t_2\rangle}\in_{(\tilde{\gamma}_1,\gamma_1)}\Im$ and $y_{\langle\tilde{t}_1,t_2\rangle}\in_{(\tilde{\gamma}_1,\gamma_1)}\Im$ but $z_{\langle\tilde{t},t_2\rangle}\overline{\in_{(\tilde{\gamma}_1,\gamma_1)}\vee q_{(\tilde{\delta}_1,\delta_2)}}\Im$ for $z\in x*y$. Which is contradiction to the hypothesis. Hence (a) and (b) are valid.

Conversely assume that there exist $x\in H$, and $\tilde{t}_1,\tilde{\delta}_1\in D(0,1],\ t_2,\delta_2\in[0,1)$ such that $x_{(\tilde{t},s)}\in_{(\tilde{\gamma}_1,\gamma_1)}\Im$, and $y_{(\tilde{t}_1,s_1)}\in_{(\tilde{\gamma}_1,\gamma_1)}\Im$. This implies that

$\tilde{\eta}_{\Im}(x)\succeq\tilde{t}\succ\tilde{\gamma}_1,\vartheta_{\Im}(x)\leq s<\gamma_1,\ \tilde{\eta}_{\Im}(y)\succeq\tilde{t}\succ\tilde{\gamma}_1,\vartheta_{\Im}(y)\leq s_1<\gamma_1.$

So

$\text{rmax}\left\{\inf_{z\in x*y}\tilde{\eta}_{\Im}(z),\tilde{\gamma}_1\right\}\succeq\text{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y),\tilde{\delta}_1\}\succeq\text{rmin}\{\tilde{t},\tilde{t}_1,\tilde{\delta}_1\},$

$s\min\left\{\sup_{z\in x*y}\vartheta_{\Im}(z),\gamma_2\right\}\leq\max\{\vartheta_{\Im}(x),\vartheta_{\Im}(y),\delta_2\}\leq\max\{s,s_1,\delta_2\}.$

We have the following two cases.

(a) If $\{\tilde{t},\tilde{t}_1\}\leq\tilde{\delta}_1$ and $\{s,s_1\}\geq\delta_2$, then

$\inf_{z\in x*y}\tilde{\eta}_{\Im}(z)\succeq\text{rmin}\{\tilde{t},\tilde{t}_1\}\succ\tilde{\gamma}_1,\ \sup_{z\in x*y}\vartheta_{\Im}(z)\leq\max\{s,s_1\}<\gamma_2.$

This implies that $z_{(\text{rmin}\{\tilde{t},\tilde{t}_1\},\max\{s,s_1\})}\in_{(\tilde{\gamma}_1,\gamma_1)}\Im.$

(b) If $\{\tilde{t},\tilde{t}_1\}>\tilde{\delta}_1$ and $\{s,s_1\}<\delta_2$, then

$\inf_{z\in x*y}\tilde{\eta}_{\Im}(z)+\text{rmax}\{\tilde{t},\tilde{t}_1\}\succ2\tilde{\delta}_1,\ \sup_{z\in x*y}\vartheta_{\Im}(z)+\min\{s,s_1\}<2\delta_2.$

This implies that $z_{(\text{rmin}\{\tilde{t},\tilde{t}_1\},\max\{\{s,s_1\}\})}q_{(\tilde{\delta}_1,\delta_2)}\Im$. Hence from both the cases we get $z_{(\min\{\tilde{t},\tilde{t}_1\},\max\{\{s,s_1\}\})}\in_{(\tilde{\gamma}_1,\gamma_1)}\vee q_{(\tilde{\delta}_1,\delta_2)}\Im$.

Thus $\Im=(\tilde{\eta}_{\Im},\vartheta_{\Im})$ is an $(\in_{\Gamma},\in_{\Gamma}\vee q_{\Delta})$-cubic H_v-ideal of S.

Theorem 3. *For an H_v-LA-semigroup, the following hold:*

(i) *A cubic set $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ of H is an $(\in,\in\vee q)$-cubic H_v-ideal of H if and only if the non-empty $(\in\vee q)$-cubic level set $[\Im]_{\in\vee q}(\tilde{t},\delta)$ is an H_v-ideal of H.*

(ii) *A cubic set $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ of H is an $(\in\vee q_{\kappa})$-cubic H_v-ideal of H if and only if the non-empty $(\in\vee q_K)$-cubic level set $[\Im]_{\in\vee q_K}(\tilde{t},\delta)$ is an H_v-ideal of H for $\tilde{t}\in D(0,\frac{\tilde{1}-\tilde{k}_1}{2}]$ and $s\in[\frac{1-k_2}{2},1)$.*

(iii) *A cubic set $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ of H is an $(\in_{\Gamma},\in_{\Gamma}\vee q_{\Delta})$-cubic H_v-ideal of H if and only if the non-empty $(\in_{\Gamma}\vee q_{\Delta})$-cubic level set $[\Im]_{\in_{\Gamma}\vee q_{\Delta}}(\tilde{t},\delta)$ is an H_v-ideal of H.*

Proof: Straightforward.

Theorem 4. *Let I be an H_v-ideal of H and $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ be a cubic set in H.*

(i) *If* $\tilde{\eta}_{\Im}(x)=\begin{cases}\succeq(0.5,0.5]\text{ if }x\in I\\(0,0]\text{ otherwise}\end{cases}$ *and* $\vartheta_{\Im}(x)=\begin{cases}\leq0.5\text{ if }x\in I\\1\text{ otherwise}\end{cases}$,

then $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ is an $(\in,\in\vee q)$-cubic H_v-ideal of H.

(ii) *If* $\tilde{\eta}_{\Im}(x)=\begin{cases}\succeq(\frac{\tilde{1}-\tilde{k}_1}{2},\frac{\tilde{1}-\tilde{k}_1}{2}]\text{ if }x\in I\\(0,0]\text{ otherwise}\end{cases}$ *and* $\vartheta_{\Im}(x)=\begin{cases}\leq\frac{1-k_2}{2}\text{ if }x\in I\\1\text{ otherwise}\end{cases}$,

then $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ is an $(\in,\in\vee q_{\kappa})$-cubic H_v-ideal of H.

(iii) *If* $\tilde{\eta}_{\Im}(x)=\begin{cases}\succeq(\delta_1,\delta_1]\text{ if }x\in I\\\succeq(\gamma_1,\gamma_1]\text{ otherwise}\end{cases}$ *and* $\vartheta_{\Im}(x)=\begin{cases}\leq\delta_2\text{ if }x\in I\\\gamma_2\text{ otherwise}\end{cases}$,

then $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ is an $(\in_{\Gamma},\in_{\Gamma}\vee q_{\Delta})$-cubic H_v-ideal of H.

Proof: (i) Let us suppose that $x,y\in H$ and let $x_{(\tilde{t}_1,t_2)}\in\Im$ where $\tilde{t}_1\in D(0,1]$ and $t_2\in[0,1)$. Then

$\tilde{\eta}_{\Im}(x)\succeq\tilde{t}_1\succ\tilde{0}$ and $\vartheta_{\Im}(x)\leq t_2<1.$

This shows that $x\in I$ and so $z\in x*y\in I$. Thus

$\tilde{\eta}_{\Im}(z)\geq(0.5,0.5]$ and $\vartheta_{\Im}(z)\leq0.5.$

Now if

$\text{rmin}\{\tilde{t}_1,\tilde{t}_3\}\preceq(0.5,0.5]$ and $\max\{t_2,t_4\}\geq0.5,$

then

$\tilde{\eta}_{\Im}(z)\succeq[0.5,0.5]\succeq\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\vartheta_{\Im}(z)\leq0.5\leq\max\{t_2,t_4\},$

for all $z\in x*y$. On the other hand if

$\text{rmin}\{\tilde{t}_1,\tilde{t}_3\}\succ[0.5,0.5]$ and $\max\{t_2,t_4\}<0.5,$

then

$\tilde{\eta}_{\Im}(z)+\text{rmin}\{\tilde{t}_1,\tilde{t}_3\}\succ[0.5,0.5]+[0.5,0.5]=\tilde{1}, \vartheta_{\Im}(z)+\max\{t_2,t_4\}<0.5+0.5=1,$

for all $z \in x*y$. Therefore $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})} \in \vee q\Im,$

for all $z \in x*y$.

Hence $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ is an $(\in,\in\vee q)$-cubic H $_v$-left ideal of H.

(ii) Straightforward.

(iii) Let us suppose that $x, y \in H$ and let $x_{(\tilde{t}_1,t_2)}q_{\Delta}\Im$ where $\tilde{t}_1,\tilde{\delta}_1 \in D(0,1]$ and $t_2,\delta_2 \in [0,1)$. Then

$\tilde{\eta}_{\Im}(x)+\tilde{t}_1 \succ 2\tilde{\delta}_1$ and $\vartheta_{\Im}(x)+t_2<2\delta_2$.

This shows that $x \in I$ and so $z \in x*y \in I$. Now if

$\tilde{\gamma}_1 \prec \text{rmin}\{\tilde{t}_1,\tilde{t}_3\}\preceq[0.5,0.5],\ \gamma_2>\max\{t_2,t_4\}\geq 0.5,$

then

$\tilde{\eta}_{\Im}(z)\succeq[0.5,0.5]\succeq\text{rmin}\{\tilde{t}_1,\tilde{t}_3\}\succ\tilde{\gamma}_1,\ \vartheta_{\Im}(z)\leq 0.5\leq\max\{t_2,t_4\}<\gamma_2,$

for all $z \in x*y$. On the other hand if

$\text{rmin}\{\tilde{t}_1,\tilde{t}_3\}\succ[0.5,0.5]\succ\tilde{\delta}_1$ and $\max\{t_2,t_4\}<0.5<\delta_2$

then

$\tilde{\eta}_{\Im}(z)+\text{rmin}\{\tilde{t}_1,\tilde{t}_3\}\succ[0.5,0.5]+[0.5,0.5]=2\tilde{\delta}_1,$

$\vartheta_{\Im}(z)+\max\{t_2,t_4\}<0.5+0.5=2\delta_2,$

for all $z \in x*y$. Therefore $z_{(\text{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})} \in_{\Gamma} \vee q_{\Delta}\Im,$

for all $z \in x*y$.

Hence $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ is an $(\in_{\Gamma},\in_{\Gamma}\vee q_{\Delta})$-cubic H $_v$-left ideal of H.

Theorem 5. *The cubic set* $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ *of* H *is an* $(\in,\in\vee q)$(*resp.,* $(\in,\in\vee q_{\kappa})$, $(\in_{\Gamma},\in_{\Gamma}\vee q_{\Delta})$)*-cubic H* $_v$*-ideal of* H. *Then the set* $\Im_{\langle\tilde{0},1\rangle}$ *is an H* $_v$*-ideal of* H.

Proof: (i) Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ be an $(\in,\in\vee q)$-cubic H $_v$-ideal of H. Let $x, y \in \Im_{\langle\tilde{0},1\rangle}$. This implies

$\tilde{\eta}_{\Im}(x)\succ\tilde{0}, \vartheta_{\Im}(x)<1,\ \tilde{\eta}_{\Im}(y)\succ\tilde{0}, \vartheta_{\Im}(y)<1,$

for $y \in H$. Let $\tilde{\eta}_{\Im}(z)=\tilde{0}$ and $\vartheta_{\Im}(z)=1$ for $z \in x*y$.

Then $x_{\langle\tilde{\eta}_{\Im}(x),\vartheta_{\Im}(x)\rangle} \in \Im$ but

$\tilde{\eta}_{\Im}(z)=\tilde{0}\prec\text{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y)\},\ \vartheta_{\Im}(z)=1>\max\{\vartheta_{\Im}(x),\vartheta_{\Im}(y)\}.$

Also

$\tilde{\eta}_{\Im}(z)+\text{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y)\}\preceq\tilde{0}+\tilde{1}=\tilde{1}, \vartheta_{\Im}(z)+\max\{\vartheta_{\Im}(x),\vartheta_{\Im}(y)\}\geq 0+1=1.$

Thus $z_{\langle\text{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\Im}(y)\},\max\{\vartheta_{\Im}(x),\vartheta_{\Im}(y)\}\rangle}\overline{\in\vee q}\Im$, a contradiction and hence $\tilde{\eta}_{\Im}(z)\succ\tilde{0}$ and $\vartheta_{\Im}(z)<1$ for $z \in x*y$. So $z \in x*y \in \Im_{\langle\tilde{0},1\rangle}$. Hence $\Im_{\langle\tilde{0},1\rangle}$ is an H $_v$-ideal of H.

(ii) and (iii) are straightforward.

4. DIRECT PRODUCT OF H $_v$-LA-SEMIGROUPS IN TERMS OF GENERALIZED CUBIC SETS

In this section we define the concept of cubic ideals and generalized cubic ideals of the direct product of H $_v$-LA-semigroups and discuss some basic properties.

Definition 15. *Let* $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ *and* $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ *be two cubic subsets of* H_1 *and* H_2 *respectively. The direct product of the cubic set* $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ *and* $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ *is defined by* $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\vartheta_{\Im\times\mathbf{F}}\rangle$, *where*

$\tilde{\eta}_{\Im\times\mathbf{F}}(x,y)=\text{rmin}\{\tilde{\eta}_{\Im}(x),\tilde{\eta}_{\mathbf{F}}(y)\}$ and $\vartheta_{\Im\times\mathbf{F}}(x,y)=\max\{\vartheta_{\Im}(x),\vartheta_{\mathbf{F}}(y)\},$

for all $(x,y)\in H_1\times H_2$.

Definition 16. *Let* $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ *and* $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ *be two cubic subsets of* H_1 *and* H_2 *respectively. The direct product* $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\vartheta_{\Im\times\mathbf{F}}\rangle$ *is called cubic H* $_v$*-LA-subsemigroup of* $H_1\times H_2$, *if*

(a)

$$\text{rinf}\ \{\tilde{\eta}_{\Im\times\mathbf{F}}(e,f)\}\succeq\text{rmin}\{\tilde{\eta}_{\Im\times\mathbf{F}}(a,b),\tilde{\eta}_{\Im\times\mathbf{F}}(c,d)\},$$

(b)

$$\sup_{(e,f)\in(a,b)*(c,d)}\{\vartheta_{\Im\times\mathbf{F}}(e,f)\}\leq\max\{\vartheta_{\Im\times\mathbf{F}}(a,b),\vartheta_{\Im\times\mathbf{F}}(c,d)\}$$

for all $(a,b),(c,d),(e,f)\in H_1\times H_2$.

Example 3. *Let* $H_1=\{a,b,c\}$ *and* $H_2=\{x,y,z,w\}$ *be two H v-LA-subsemigroups with the following table:*

$*$	a	b	c
a	a	$\{a,c\}$	H
b	$\{a,c\}$	a	a
c	$\{a,b\}$	c	$\{a,c\}$

$\bullet$	x	y	z	w
x	$\{x,w\}$	$\{x,y,w\}$	y	$\{x,w\}$
y	$\{x,y,w\}$	$\{y,w\}$	x	$\{y,w\}$
z	x	y	z	w
w	$\{x,w\}$	$\{y,w\}$	w	w

Then

$H_1\times H_2=\{(a,x),(a,y),(a,z),(a,w),(b,x),(b,y),(b,z),(b,w),(c,x),(c,y),(c,z),(c,w)\}$

is an H v-LA-semigroup with the hyperoperation defined as

$(a_1,b_1)\otimes(a_2,b_2)=\{(c,d)\mid c\in a_1*a_2,\ d\in b_1\bullet b_2\},$

for all $(a_1,b_1),(a_2,b_2)\in H_1\times H_2$. *Define the cubic set* $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\vartheta_{\Im\times\mathbf{F}}\rangle$ *as*

$\tilde{\eta}_{\Im\times\mathbf{F}}(a,x)=\tilde{\eta}_{\Im\times\mathbf{F}}(a,y)=\tilde{\eta}_{\Im\times\mathbf{F}}(a,z)=\tilde{\eta}_{\Im\times\mathbf{F}}(a,w)=\tilde{\eta}_{\Im\times\mathbf{F}}(b,x)=\tilde{\eta}_{\Im\times\mathbf{F}}(c,x)=(0.6,0.7],$

$\tilde{\eta}_{\Im\times\mathbf{F}}(b,y)=\tilde{\eta}_{\Im\times\mathbf{F}}(b,z)=\tilde{\eta}_{\Im\times\mathbf{F}}(b,w)=\tilde{\eta}_{\Im\times\mathbf{F}}(c,y)=\tilde{\eta}_{\Im\times\mathbf{F}}(c,z)=\tilde{\eta}_{\Im\times\mathbf{F}}(c,w)=(0.7,0.8]$

and

$\vartheta_{\Im\times\mathbf{F}}(a,x)=\vartheta_{\Im\times\mathbf{F}}(a,y)=\vartheta_{\Im\times\mathbf{F}}(a,z)=\vartheta_{\Im\times\mathbf{F}}(a,w)=\vartheta_{\Im\times\mathbf{F}}(b,x)=\vartheta_{\Im\times\mathbf{F}}(c,x)=0.5,$

$\vartheta_{\Im\times\mathbf{F}}(b,y)=\vartheta_{\Im\times\mathbf{F}}(b,z)=\vartheta_{\Im\times\mathbf{F}}(b,w)=\vartheta_{\Im\times\mathbf{F}}(c,y)=\vartheta_{\Im\times\mathbf{F}}(c,z)=\vartheta_{\Im\times\mathbf{F}}(c,w)=0.4.$

Then it is clear that $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\eta_{\Im\times\mathbf{F}}\rangle$ *is cubic H v-LA-subsemigroup of* $H_1\times H_2$.

Definition 17. *Let* $\mathfrak{I}=\langle\tilde{\eta}_{\mathfrak{I}},\vartheta_{\mathfrak{I}}\rangle$ *and* $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ *be cubic subset of* H_1 *and* H_2 *respectively. Then direct product* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *is called a cubic H v-left (resp., H v-right) ideal of* $H_1\times H_2$, *if*

(a) $\operatorname{rinf}_{(e,f)\in(a,b)*(c,d)}\{\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}}(e,f)\}\succeq\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}}(c,d)$ *(resp.,* $\operatorname{rinf}_{(e,f)\in(a,b)*(c,d)}\{\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}}(e,f)\}\succeq\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}}(a,b)$ *)*

(b) $\sup_{(e,f)\in(a,b)*(c,d)}\{\vartheta_{\mathfrak{I}\times\mathbf{F}}(e,f)\}\leq\vartheta_{\mathfrak{I}\times\mathbf{F}}(c,d)$ *(resp.,* $\sup_{(e,f)\in(a,b)*(c,d)}\{\vartheta_{\mathfrak{I}\times\mathbf{F}}(e,f)\}\leq\vartheta_{\mathfrak{I}\times\mathbf{F}}(a,b)$ *)*

for all $(a,b),(c,d)\in H_1\times H_2$.

Definition 18. *The direct product* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *is called* $(\in,\in\vee q)$*-cubic* H_v*-LA-subsemigroup of* $H_1\times H_2$, *if it satisfies,*

(DS_1) $(a,b)_{(\tilde{t}_1,t_2)}\in\mathfrak{I}\times\mathbf{F}$ *and* $(c,d)_{(\tilde{t}_3,t_4)}\in\mathfrak{I}\times\mathbf{F}$ *imply that* $(e,f)_{(\operatorname{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in\vee q\mathfrak{I}\times\mathbf{F}$, *for all* $(e,f)\in(a,b)*(c,d)$.

Definition 19. *The direct product* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *is called* $(\in,\in\vee q_K)$*-cubic* H_v*-LA-subsemigroup of* $H_1\times H_2$, *if it satisfies,*

(DS_2) $(a,b)_{(\tilde{t}_1,t_2)}\in\mathfrak{I}\times\mathbf{F}$ *and* $(c,d)_{(\tilde{t}_3,t_4)}\in\mathfrak{I}\times\mathbf{F}$ *imply that* $(e,f)_{(\operatorname{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in\vee q_K\mathfrak{I}\times\mathbf{F}$, *for all* $(e,f)\in(a,b)*(c,d)$.

Definition 20. *The direct product* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *is called* $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$*-cubic* H_v*-LA-subsemigroup of* $H_1\times H_2$, *if it satisfies,* (DS_3) $(a,b)_{(\tilde{t}_1,t_2)}\in_\Gamma\mathfrak{I}\times\mathbf{F}$ *and* $(c,d)_{(\tilde{t}_3,t_4)}\in_\Gamma\mathfrak{I}\times\mathbf{F}$ *imply that* $(e,f)_{(\operatorname{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in_\Gamma\vee q_\Delta\mathfrak{I}$, *for all* $(e,f)\in(a,b)*(c,d)$.

Definition 21. *The direct product* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *is called* $(\in,\in\vee q)$*-cubic H v-left (resp., H v-right)ideal of* $H_1\times H_2$, *if it satisfies,* (DI_1) $(c,d)_{(\tilde{t}_3,t_4)}\in\mathfrak{I}$ *and* $(a,b)\in H_1\times H_2$ *imply that* $(e,f)_{(\operatorname{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in\vee q\mathfrak{I}$, *for all* $(e,f)\in(a,b)*(c,d)$ *(resp.,* $(e,f)\in(c,d)*(a,b)$*).*

Definition 22. *The direct product* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *is called* $(\in,\in\vee q_K)$*-cubic H v-left (resp., H v-right) ideal of* $H_1\times H_2$, *if it satisfies,* (DI_2) $(c,d)_{(\tilde{t}_3,t_4)}\in\mathfrak{I}$ *and* $(a,b)\in H_1\times H_2$ *imply that* $(e,f)_{(\operatorname{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in\vee q_K\mathfrak{I}$, *for all* $(e,f)\in(a,b)*(c,d)$ *(resp.,* $(e,f)\in(c,d)*(a,b)$*).*

Definition 23. *The direct product* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *is called* $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$*-cubic H v-left (resp., H v-right) ideal of* $H_1\times H_2$, *if it satisfies,* (DI_3) $(c,d)_{(\tilde{t}_3,t_4)}\in_\Gamma\mathfrak{I}$ *and* $(a,b)\in H_1\times H_2$ *imply that* $(e,f)_{(\operatorname{rmin}\{\tilde{t}_1,\tilde{t}_3\},\max\{t_2,t_4\})}\in_\Gamma\vee q_\Delta\mathfrak{I}$, *for all* $(e,f)\in(a,b)*(c,d)$ *(resp.,* $(e,f)\in(c,d)*(a,b)$*).*

Theorem 6. *For the direct product of two* H_v*-LA-semigroups, the following hold:*

(i) *A cubic set* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *of* $H_1\times H_2$ *is an* $(\in,\in\vee q)$*-cubic* H_v*-LA-subsemigroup (resp., cubic* H_v*-ideal) of* $H_1\times H_2$ *if and only if*

(a) $\left\{\operatorname{rinf}_{(e,f)\in(a,b)*(c,d)}\{\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}}(e,f)\}\right\}\succeq\operatorname{rmin}\{\operatorname{rmax}\{\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}}(a,b),\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}}(c,d)\},(0.5,0.5]\}$,

(b) $\left\{\sup_{(e,f)\in(a,b)*(c,d)}\vartheta_{\mathfrak{I}\times\mathbf{F}}(e,f)\right\}\leq\max\{\min\{\vartheta_{\mathfrak{I}\times\mathbf{F}}(a,b),\vartheta_{\mathfrak{I}\times\mathbf{F}}(c,d)\},0.5\}$.

(ii) *A cubic set* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *of* $H_1\times H_2$ *is an* $(\in,\in\vee q_K)$*-cubic* H_v*-LA-subsemigroup (resp., cubic* H_v*-ideal) of* $H_1\times H_2$ *if and only if*

(a) $\left\{\operatorname{rinf}\{\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}}(e,f)\}\right\}\succeq\operatorname{rmin}\{\operatorname{rmax}\{\tilde{\eta}(a,b),\tilde{\eta}(c,d)\},\frac{1-\tilde{k}_1}{2}\}$,

(b) $\left\{\sup_{(e,f)\in(a,b)*(c,d)}\vartheta_{\mathfrak{I}\times\mathbf{F}}(e,f)\right\}\leq\max\{\min\{\vartheta_{\mathfrak{I}\times\mathbf{F}}(a,b),\vartheta_{\mathfrak{I}\times\mathbf{F}}(c,d)\},\frac{1-k_2}{2}\}$.

(iii) *A cubic set* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *of* $H_1\times H_2$ *is an* $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$*-cubic* H_v*-LA-subsemigroup (resp., cubic* H_v*-ideal) of* $H_1\times H_2$ *if and only if*

(a) $\operatorname{rmax}\left\{\operatorname{rinf}\{\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}}(e,f)\},\tilde{\gamma}_1\right\}\succeq\operatorname{rmin}\{\operatorname{rmax}\{\tilde{\eta}(a,b),\tilde{\eta}(c,d)\},\tilde{\delta}_1\}$,

(b) $\min\left\{\sup_{(e,f)\in(a,b)*(c,d)}\vartheta_{\mathfrak{I}\times\mathbf{F}}(e,f),\gamma_2\right\}\leq\max\{\min\{\vartheta_{\mathfrak{I}\times\mathbf{F}}(a,b),\vartheta_{\mathfrak{I}\times\mathbf{F}}(c,d)\},\delta_2\}$.

Proof: Straightforward.

Theorem 6. *Let* $\mathfrak{I}=\langle\tilde{\eta}_{\mathfrak{I}},\vartheta_{\mathfrak{I}}\rangle$ *and* $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ *be any two cubic* H_v*-LA-subsemigroups (resp., cubic* H_v*-ideals) of* H_1 *and* H_2 *respectively. Then the direct product* $\mathfrak{I}\times\mathbf{F}=\langle\tilde{\eta}_{\mathfrak{I}\times\mathbf{F}},\vartheta_{\mathfrak{I}\times\mathbf{F}}\rangle$ *is cubic* H_v*-LA-subsemigroup (resp., cubic* H_v*-ideal) of* $H_1 \times H_2$.

Proof: Let $\mathfrak{I}=\langle\tilde{\eta}_{\mathfrak{I}},\vartheta_{\mathfrak{I}}\rangle$ and $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ be any two cubic H_v-LA-subsemigroups of H_1 and H_2 respectively. For any $(a,b),(c,d)\in H_1\times H_2$, we have

$$\operatorname{rinf}_{(e,f)\in(a,b)*(c,d)}\{\tilde{\eta}_{\Im\times\mathbf{F}}(e,f)\}=\operatorname{rinf}_{e\in a*c}\tilde{\eta}_{\Im}(e)\operatorname{rmin}\operatorname{rinf}_{f\in b*d}\tilde{\eta}_{\mathbf{F}}(f)$$
$$\succeq\operatorname{rmin}\{\tilde{\eta}_{\Im}(a),\tilde{\eta}_{\Im}(c)\}\operatorname{rmin}\operatorname{rmin}\{\tilde{\eta}_{\mathbf{F}}(b),\tilde{\eta}_{\mathbf{F}}(d)\}$$
$$=\operatorname{rmin}\{\tilde{\eta}_{\Im}(a)\operatorname{rmin}\tilde{\eta}_{\mathbf{F}}(b),\tilde{\eta}_{\Im}(c)\operatorname{rmin}\tilde{\eta}_{\mathbf{F}}(d)\}$$
$$=\operatorname{rmin}\{\tilde{\eta}_{\Im\times\mathbf{F}}(a,b),\tilde{\eta}_{\Im\times\mathbf{F}}(c,d)\}$$

and

$$\sup_{(e,f)\in(a,b)*(c,d)}\{\vartheta_{\Im\times\mathbf{F}}(e,f)\}=\sup_{e\in a*c}\vartheta_{\Im}(e)\vee\sup_{f\in b*d}\vartheta_{\mathbf{F}}(f)$$
$$\leq\sup\{\vartheta_{\Im}(a),\vartheta_{\Im}(c)\}\vee\sup\{\vartheta_{\mathbf{F}}(b),\vartheta_{\mathbf{F}}(d)\}$$
$$=\sup\{\vartheta_{\Im}(a)\vee\vartheta_{\mathbf{F}}(b),\vartheta_{\Im}(c)\vee\vartheta_{\mathbf{F}}(d)\}$$
$$=\sup\{\vartheta_{\Im\times\mathbf{F}}(a,b),\vartheta_{\Im\times\mathbf{F}}(c,d)\}.$$

Hence $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\vartheta_{\Im\times\mathbf{F}}\rangle$ is cubic H $_v$-LA-subsemigroup of $H_1\times H_2$. Similarly other case can be proved.

Theorem 7. Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ and $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ be any two $(\in,\in\vee q)$ (resp., $(\in,\in\vee q_K),(\in_\Gamma,\in_\Gamma\vee q_\Delta)$) cubic H $_v$-LA-subsemigroups (resp., cubic H $_v$-ideals) of H_1 and H_2 respectively. Then the direct product $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\vartheta_{\Im\times\mathbf{F}}\rangle$ *is an* $(\in,\in\vee q)$ *(resp.,* $(\in,\in\vee q_K),(\in_\Gamma,\in_\Gamma\vee q_\Delta)$ *)-cubic H* $_v$*-LA-subsemigroup (resp., cubic H* $_v$ *-ideal) of* $H_1\times H_2$.

Proof: Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ and $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ be any two $(\in,\in\vee q)$-cubic H $_v$-LA-subsemigroups of H_1 and H_2 respectively. For any $(a,b),(c,d)\in H_1\times H_2$, we have

$$\operatorname{rinf}_{(e,f)\in(a,b)*(c,d)}\{\tilde{\eta}_{\Im\times\mathbf{F}}(e,f)\}=\operatorname{rinf}_{e\in a*c}\{\tilde{\eta}_{\Im}(e)\}\operatorname{rmin}\operatorname{rinf}_{f\in b*d}\{\tilde{\eta}_{\mathbf{F}}(f)\}$$
$$\succeq\operatorname{rmin}\{\operatorname{rmax}[\tilde{\eta}_{\Im}(a),\tilde{\eta}_{\Im}(c)],(0.5,0.5]\}$$
$$\operatorname{rmin}\operatorname{rmin}\{\operatorname{rmax}[\tilde{\eta}_{\mathbf{F}}(b),\tilde{\eta}_{\mathbf{F}}(d)],(0.5,0.5]\}$$
$$=\operatorname{rmin}\{\operatorname{rmax}[\tilde{\eta}_{\Im}(a)\operatorname{rmin}\tilde{\eta}_{\mathbf{F}}(b),\tilde{\eta}_{\Im}(c)\operatorname{rmin}\tilde{\eta}_{\mathbf{F}}(d)],(0.$$
$$=\operatorname{rmin}\{\operatorname{rmax}[\tilde{\eta}_{\Im\times\mathbf{F}}(a,b),\tilde{\eta}_{\Im\times\mathbf{F}}(c,d)],(0.5,0.5]\}$$

and

$$\sup_{(e,f)\in(a,b)*(c,d)}\{\vartheta_{\Im\times\mathbf{F}}(e,f)\}=\sup_{e\in a*c}\{\vartheta_{\Im}(e)\}\vee\sup_{f\in b*d}\{\vartheta_{\mathbf{F}}(f)\}$$
$$\leq\max\{\min[\vartheta_{\Im}(a),\vartheta_{\Im}(c)],0.5\}$$
$$\vee\max\{\min[\vartheta_{\mathbf{F}}(b),\vartheta_{\mathbf{F}}(d)],0.5\}$$
$$=\max\{\min[\vartheta_{\Im}(a)\vee\vartheta_{\mathbf{F}}(b),\vartheta_{\Im}(c)\vee\vartheta_{\mathbf{F}}(d)],0.5$$
$$=\max\{\min[\vartheta_{\Im\times\mathbf{F}}(a,b),\vartheta_{\Im\times\mathbf{F}}(c,d)],0.5\}.$$

Hence $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\vartheta_{\Im\times\mathbf{F}}\rangle$ is an $(\in,\in\vee q)$-cubic H $_v$-LA-subsemigroup of $H_1\times H_2$.

Similarly it can be shown for $(\in,\in\vee q_K)$-cubic H $_v$-LA-subsemigroups and $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$-cubic H $_v$-LA-subsemigroups of $H_1\times H_2$.

Theorem 8. *Let* $\Im_1\times\mathbf{F}_1=\langle\tilde{\eta}_{\Im_1\times\mathbf{F}_1},\vartheta_{\Im_1\times\mathbf{F}_1}\rangle$ *and* $\Im_2\times\mathbf{F}_2=\langle\tilde{\eta}_{\Im_2\times\mathbf{F}_2},\vartheta_{\Im_2\times\mathbf{F}_2}\rangle$ *be any two cubic H* $_v$*-LA-subsemigroups (resp., cubic H* $_v$*-ideals) of* $H_1\times H_2$ *and* $H_1\times H_2$ *respectively. Then* $\Im_1\times\mathbf{F}_1\cap\Im_2\times\mathbf{F}_2$ *is cubic H* $_v$*-LA-subsemigroup (resp., cubic H* $_v$*-ideal) of* $H_1\times H_2$.

Proof: Straightforward.

Theorem 9. *Let* $\Im_1\times\mathbf{F}_1=\langle\tilde{\eta}_{\Im_1\times\mathbf{F}_1},\vartheta_{\Im_1\times\mathbf{F}_1}\rangle$ *and* $\Im_2\times\mathbf{F}_2=\langle\tilde{\eta}_{\Im_2\times\mathbf{F}_2},\vartheta_{\Im_2\times\mathbf{F}_2}\rangle$ *be any two* $(\in,\in\vee q)$ *(resp.,* $(\in,\in\vee q_K),(\in_\Gamma,\in_\Gamma\vee q_\Delta)$ *)-cubic H* $_v$*-LA-subsemigroups* (resp., $(\in,\in\vee q_K),(\in_\Gamma,\in_\Gamma\vee q_\Delta)$ cubic H $_v$-ideals) *of* $H_1\cdot H_2$. *Then* $f_1\cdot\mathrm{F}_1\varsigma f_2\cdot\mathrm{F}_2$ *is an* $(\in,\in\vee q)$ *(resp.,* $(\in,\in\vee q_K),(\in_\Gamma,\in_\Gamma\vee q_\Delta)$ *)-cubic H* $_v$*-LA-subsemigroups* (resp., $(\in,\in\vee q_K),(\in_\Gamma,\in_\Gamma\vee q_\Delta)$ cubic H $_v$-ideals) *of* $H_1\times H_2$.

Proof: Straightforward.

Definition 24. Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ and $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ be cubic subset of H_1 and H_2 respectively. For any $\tilde{t}\in D(0,1]$ and $s\in[0,1)$ we define the set $U(\Im\times\mathbf{F};\tilde{t},s)=\{(x,y)\in H_1\times H_2\ :\ \tilde{\eta}_{\Im\times\mathbf{F}}(x,y)\succeq\tilde{t},\vartheta_{\Im\times\mathbf{F}}(x,y)\leq s\}$, is called the cubic level set of $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\vartheta_{\Im\times\mathbf{F}}\rangle$.

Theorem 10. (i) Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ and $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ be any two cubic H $_v$-LA-subsemigroups (resp., cubic H $_v$-ideals) of H_1 and H_2 respectively. The direct product $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\vartheta_{\Im\times\mathbf{F}}\rangle$ is cubic H $_v$-LA-subsemigroup (resp., cubic H $_v$-ideals) of $H_1\times H_2$ if and only if the non-empty level set $U(\Im\times\mathbf{F};\tilde{t},s)$ is H $_v$-LA-subsemigroup(resp., cubic H $_v$ -ideal) of $H_1\times H_2$.

(ii) Let $\Im=\langle\tilde{\eta}_{\Im},\vartheta_{\Im}\rangle$ and $\mathbf{F}=\langle\tilde{\eta}_{\mathbf{F}},\vartheta_{\mathbf{F}}\rangle$ be any two $(\in,\in\vee q)$ (resp., $(\in,\in\vee q_K),(\in_\Gamma,\in_\Gamma\vee q_\Delta)$)-cubic H $_v$-LA-subsemigroups (resp., cubic H $_v$-ideals) of H_1 and H_2 respectively. Then the direct product $\Im\times\mathbf{F}=\langle\tilde{\eta}_{\Im\times\mathbf{F}},\vartheta_{\Im\times\mathbf{F}}\rangle$ is $(\in,\in\vee q)$ (resp., $(\in,\in\vee q_K),(\in_\Gamma,\in_\Gamma\vee q_\Delta)$)-cubic H $_v$-LA-subsemigroup (resp., cubic H $_v$-ideals) of $H_1\times H_2$ if and only if the non-empty level set $U(\Im\times\mathbf{F};\tilde{t},s)$ is H $_v$-LA-subsemigroup (resp., cubic H $_v$-ideal) of $H_1\times H_2$.

Proof: Straightforward.

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