HAMILTONIAN FORMULATION FOR SINGLE FLUID FLOWS WITH CAPUTO'S DEFINITION

Emad K. Jaradat[1]
[1]Department of Physics, Mu'tah University, Al-Karak, Jordan
emad_jaradat75@yahoo.com

ABSTRACT: *We reformulated the Lagrangian density for single fluid by using Caputo's fractional derivative, then from the fractional Euler-Lagrangian equation we obtained the equations that described the motion of single fluid in fractional form. Then the Hamiltonian density and the energy-stress tensor are obtained in fractional form from the fluid Lagrangian density. From the Hamiltonian density we also obtained the Hamiltonian equations of motion for the single fluid in fractional form.*

KEYWORDS: Single Fluid Lagrangian Density, Caputo's Definition, Fractional Hamiltonian.

1. INTRODUCTION

Fractional calculus appeared in many science and engineering fields, and recently has become a widely used, because the studies proved that the fractional derivatives and integrals are appropriate to solve many problems. Many applications are found in different fields, such as; dynamical systems, electrical circuits with fractance, electrochemistry, multipoles in electromagnetism, neurons in biology, the problems of viscoelasticity which be solved by Caputo, and many others[1].

In fluid field, Bernard [2] described the equations of motion for six velocity potentials of perfect fluids which led to a variational principle that reproduced the Eulerian equation of motion. Saarloos[3] shown that the density function (mass , momentum and energy fields) obeys a Liouville equation for hydrodynamics ideal fluid. Ambrosi [4] given the Hamiltonain formulation subject to the gravity force for two different irrotational isoentropic fluids density and evaluated the momentum potential density and canonical variables. Bokhove, derived the geometric link between the parcel Eulerian – Lagrangian formulation and well-known variational and Hamiltonian formulation for three models of ideal and geophysical fluid flow. Manoff [5] applied the method of lagrangian with covariant derivative to special type of lagrangian density depending on scalar and vector fields . the corresponding Euler-Lagrange's equation and energy – momentum tensors are found on the basis of the covariant Noether's identities. In another paper Manoff[6] found the Euler –Lagrange equations from an unconstrained variation principle by using the method of Lagrangian with covariant derivatives and additional conditions for the perfect fluid. Bhat [7] derived the lagrangian averaged Euler equations for barotropic compressible flows by adding dispersion instead of artificial viscosity.

The main goal of this work is to derive the equation of motion in fractional form for the single fluid from the fractional Lagrangian density and from the fractional Hamiltonian density also to determine the energy-stress tensor in fractional form also, by using Caputo's fractional derivative.

2. BASIC DEFINTIONS

In fractional calculus many definitions of derivative: Riemann-Liouvelle, Caputo, Marchaud and Riesz fractional derivatives. The definition of right and left Caputo fractional derivatives, respectively, are [8,9]

$${}_t^cD_b^\alpha = \frac{1}{\Gamma(n-\alpha)}\int_t^b(\tau-t)^{n-\alpha-1}\left(-\frac{d}{d\tau}\right)^n f(\tau)d\tau \quad (1)$$

$${}_a^cD_t^\alpha = \frac{1}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-\alpha-1}\left(\frac{d}{d\tau}\right)^n f(\tau)d\tau \quad (2)$$

Where α $(\alpha \in +\mathbb{R})$ is the order of derivative, $-1 \leq \alpha < n$, $n$is an intege. $(a, b \in \mathbb{R})$, (Γ) denotes to Euler's Gamma function.

3. LAGRANGIAN DENSITY OF SINGLE FLUID

Invicid fluid means that the force emerging from viscosity is small relative to other forces so it can be neglected [10, 11]. And the Lagrangian density for an Invicid single fluid is

$$\mathcal{L} = -\frac{1}{2}\frac{\dot{\varphi}^2}{\acute{\varphi}} + \mu \quad (3)$$

where $\varphi(i,t)$ is the potential field

$$\dot{\varphi} = \partial_0\varphi$$
$$\acute{\varphi} = \partial_i\varphi = \nabla\varphi$$

μ is chemical potential

From the variation of the action integral we can derive the Euler-Lagrangian equation[12]

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\right) = 0 \quad (4)$$

Apply this equation for the Lagrangian density for an Invicid single fluid

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\varphi_{\mathrm{r},\alpha}}\right) = \partial_0\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\right) + \partial_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}\right)$$

$$= -\partial_0\frac{\partial_0\varphi}{\partial_i\varphi} + \frac{1}{2}\partial_i\left(\frac{\partial_0\varphi}{\partial_i\varphi}\right)^2 + \partial_i\mu \quad (5)$$

Substitute the results in Eq.(4) , we get the equation of motion

$$\partial_0\frac{\partial_0\varphi}{\partial_i\varphi} - \frac{1}{2}\partial_i\left(\frac{\partial_0\varphi}{\partial_i\varphi}\right)^2 - \partial_i\mu = 0 \quad (6)$$

The Hamiltonian density is

$$\mathcal{H} = \pi_r(i,t)\dot{\varphi}_r(i,t) - \mathcal{L}\left(\varphi_\mathrm{r}, \varphi_{\mathrm{r},\alpha}\right) \quad (7)$$

where $\pi_r(i,t)$ is the conjugate momentum and can be determined from the relation

$$\pi_r(i,t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)} = -\frac{\partial_0\varphi}{\partial_i\varphi}$$

Then the Hamiltonian density becomes

$$\mathcal{H} = -\frac{1}{2}\frac{(\partial_0\varphi)^2}{\partial_i\varphi} - \mu\partial_i\varphi \quad (8)$$

in terms of the conjugate momentum(π), the Hamiltonian density is

$$\mathcal{H} = -\left(\frac{1}{2}\pi^2 + \mu\right)\partial_i\varphi \quad (9)$$

The Hamiltonian equation of motion is

$$\frac{\partial\mathcal{H}}{\partial\varphi} = -\dot{\pi} - \partial_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}\right)$$

$$\frac{\partial\mathcal{H}}{\partial\varphi} = 0$$

$0=-\dot{\pi}-\frac{1}{2}\partial_i(\frac{\partial_0\varphi}{\partial_i\varphi})^2-\partial_i\mu$(10)

The time derivative of conjugate momentum is

$\dot{\pi}\ =\ -\frac{{\partial_0}^2\varphi}{\partial_i\varphi}$(11)

Substitute Eq.(11) in Eq.(10), we get the Hamiltonian equation of motion which is the same result of Lagrangian equation of motion.

$0=\frac{{\partial_0}^2\varphi}{\partial_i\varphi}-\frac{1}{2}\partial_i(\frac{\partial_0\varphi}{\partial_i\varphi})^2-\partial_i\mu$(12)

To calculate the energy- stress tensor, apply the following relation which can be derived from the Noether's Theorem [13]

$\mathrm{T}_\nu^\mu=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi-\delta_\nu^\mu\mathcal{L}$(13)

$T_0^0=\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\partial_0\varphi-\mathcal{L}=-\frac{\partial_0\varphi}{\partial_i\varphi}\partial_0\varphi+\frac{1}{2}\frac{(\partial_0\varphi)^2}{\partial_i\varphi}-\mu\partial_i\varphi$(14)

$T_0^0=-\frac{1}{2}\frac{(\partial_0\varphi)^2}{\partial_i\varphi}-\mu\partial_i\varphi$(15)

Notice, T_0^0 equale to the Hamiltonian density, $T_0^0=\mathcal{H}$

$T_i^0=\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\partial_i\varphi=-\frac{\partial_0\varphi}{\partial_i\varphi}\partial_i\varphi=-\partial_0\varphi$(16)

which represents to the momentum density.

$T_0^i=\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}\partial_0\varphi=\left(\frac{1(\partial_0\varphi)^2}{2(\partial_i\varphi)^2}+\mu\right)\partial_0\varphi\ =\ \frac{1(\partial_0\varphi)^3}{2(\partial_i\varphi)^2}+\mu\partial_0\varphi$(17)

which represents to the density of momentum component i.

for $\mu\neq\nu\neq 0$

$T_j^i=\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}\partial_j\varphi=\left(\frac{1(\partial_0\varphi)^2}{2(\partial_i\varphi)^2}+\mu\right)\partial_j\varphi$(18)

for $\mu=\nu\neq 0$

$T_i^i=\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}\partial_i\varphi-\mathcal{L}=\left(\frac{1(\partial_0\varphi)^2}{2(\partial_i\varphi)^2}+\mu\right)\partial_i\varphi+\frac{1}{2}\frac{(\partial_0\varphi)^2}{\partial_i\varphi}-\mu\partial_i\varphi$(19)

$T_i^i=\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)}\partial_i\varphi-\mathcal{L}=\frac{1(\partial_0\varphi)^2}{2\ \partial_i\varphi}+\mu\partial_i\varphi+\frac{1}{2}\frac{(\partial_0\varphi)^2}{\partial_i\varphi}-\mu\partial_i\varphi$(20)

4. EQUATIONS OF MATION IN FRACTIONAL FORM FOR SINGLE FLUID LAGRANGIAN DENSITY

The fractional counterpart for this Lagrangian density is

$\mathcal{L}=-\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{{}_a^cD_{x_i}^\alpha\varphi}+\mu{}_a^cD_{x_i}^\alpha\varphi$(21)

the equation of motion using the Euler-Lagrangian equation in fractional form is

$$\frac{\partial\mathcal{L}}{\partial\varphi}+{}_t^cD_b^\alpha\frac{\partial\mathcal{L}}{\partial{}_a^cD_t^\alpha\varphi}+{}_{x_i}^cD_b^\alpha\frac{\partial\mathcal{L}}{\partial{}_a^cD_{x_i}^\alpha\varphi}+{}_a^cD_t^\beta\frac{\partial\mathcal{L}}{\partial{}_t^cD_b^\beta\varphi}+{}_a^cD_{x_i}^\beta\frac{\partial\mathcal{L}}{\partial{}_{x_i}^cD_b^\beta\varphi}=0\quad(22)$$

$$\frac{\partial\mathcal{L}}{\partial\varphi}=0,\ \ {}_a^cD_t^\beta\frac{\partial\mathcal{L}}{\partial{}_t^cD_b^\beta\varphi}0\ \ ,\ \ {}_a^cD_{x_i}^\beta\frac{\partial\mathcal{L}}{\partial{}_{x_i}^cD_b^\beta\varphi}=0$$

$${}_t^cD_b^\alpha\frac{\partial\mathcal{L}}{\partial{}_a^cD_t^\alpha\varphi}={}_t^cD_b^\alpha(-\frac{{}_a^cD_t^\alpha\varphi}{{}_a^cD_{x_i}^\alpha\varphi})$$

$${}_{x_i}^cD_b^\alpha\frac{\partial\mathcal{L}}{\partial{}_a^cD_{x_i}^\alpha\varphi}={}_{x_i}^cD_b^\alpha\left(\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{\left({}_a^cD_{x_i}^\alpha\varphi\right)^2}+\mu\right)$$

Eq.(20) becomes

${}_t^cD_b^\alpha\left(-\frac{{}_a^cD_t^\alpha\varphi}{{}_a^cD_{x_i}^\alpha\varphi}\right)+{}_{x_i}^cD_b^\alpha\left(\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{\left({}_a^cD_{x_i}^\alpha\varphi\right)^2}+\mu\right)=0$(23)

use ${}_t^cD_b^\alpha=-{}_a^cD_t^\alpha$, ${}_{x_i}^cD_b^\alpha=-{}_a^cD_{x_i}^\alpha$

$${}_a^cD_t^\alpha\left(\frac{{}_a^cD_t^\alpha\varphi}{{}_a^cD_{x_i}^\alpha\varphi}\right)-{}_a^cD_{x_i}^\alpha\left(\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{\left({}_a^cD_{x_i}^\alpha\varphi\right)^2}+\mu\right)=0$$

(24)

$\frac{({}_a^cD_t^\alpha)^2\varphi}{{}_a^cD_{x_i}^\alpha\varphi}-\frac{1}{2}{}_a^cD_{x_i}^\alpha\left(\frac{{}_a^cD_t^\alpha\varphi}{{}_a^cD_{x_i}^\alpha\varphi}\right)^2-{}_a^cD_{x_i}^\alpha\mu$ (25)

The expression of the fractional conjugate momentum is

$$\pi_\alpha=\frac{\partial\mathcal{L}}{\partial({}_a^cD_t^\alpha\varphi)}$$

For the invicid single fluid

$$(\pi_\alpha)_\varphi=-\frac{{}_a^cD_t^\alpha\varphi}{{}_a^cD_{x_i}^\alpha\varphi}$$

The fractional Hamiltonian density is

$\mathcal{H}\ =\ \pi_\alpha({}_a^cD_t^\alpha\varphi)-\mathcal{L}$(26)

$\mathcal{H}=\left(-\frac{{}_a^cD_t^\alpha\varphi}{{}_a^cD_{x_i}^\alpha\varphi}\right)({}_a^cD_t^\alpha\varphi)+\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{{}_a^cD_{x_i}^\alpha\varphi}-\mu{}_a^cD_{x_i}^\alpha\varphi$(27)

$$\mathcal{H}=\left(-\frac{({}_a^cD_t^\alpha\varphi)^2}{{}_a^cD_{x_i}^\alpha\varphi}\right)+\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{{}_a^cD_{x_i}^\alpha\varphi}-\mu{}_a^cD_{x_i}^\alpha\varphi$$

(28)

$\mathcal{H}=\left(-\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{{}_a^cD_{x_i}^\alpha\varphi}-\mu{}_a^cD_{x_i}^\alpha\varphi\right)$(29)

The fractional Hamiltonian equation of motion can be determined as follow

$\frac{\partial\mathcal{H}}{\partial\varphi}={}_t^cD_b^\alpha\pi_\alpha+{}_a^cD_t^\beta\pi_\beta+{}_{x_i}^cD_b^\alpha\frac{\partial\mathcal{L}}{\partial{}_a^cD_{x_i}^\alpha\varphi}+{}_a^cD_{x_i}^\beta\frac{\partial\mathcal{L}}{\partial{}_{x_i}^cD_b^\beta\varphi}$(30)

$$\frac{\partial\mathcal{H}}{\partial\varphi}=0\ ,\qquad\pi_\beta=0\ ,\qquad\frac{\partial\mathcal{L}}{\partial{}_{x_i}^cD_b^\beta\varphi}=0$$

$$\frac{\partial\mathcal{L}}{\partial{}_a^cD_{x_i}^\alpha\varphi}=\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{\left({}_a^cD_{x_i}^\alpha\varphi\right)^2}+\mu$$

Put these results in Eq.(22) and use($-{}_t^cD_b^\alpha={}_a^cD_t^\alpha$) and (${}_{x_i}^cD_b^\alpha=-{}_a^cD_{x_i}^\alpha$) we have

$$0={}_t^cD_b^\alpha\left(-\frac{{}_a^cD_t^\alpha\varphi}{{}_a^cD_{x_i}^\alpha\varphi}\right)+{}_{x_i}^cD_b^\alpha\left(\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{\left({}_a^cD_{x_i}^\alpha\varphi\right)^2}+\mu\right)$$

$$0={}_a^cD_t^\alpha\left(\frac{{}_a^cD_t^\alpha\varphi}{{}_a^cD_{x_i}^\alpha\varphi}\right)-{}_a^cD_{x_i}^\alpha\left(\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{\left({}_a^cD_{x_i}^\alpha\varphi\right)^2}+\mu\right)$$

which is the same with Euler-Lagrangian equation of motion.

The fractional energy stress tensor is

$T_\nu^\mu=\frac{\partial\mathcal{L}}{\partial{}_a^cD_{x_\mu}^\alpha\phi}{}_a^cD_{x_\nu}^\alpha\phi-\delta_\nu^\mu\mathcal{L}$(31)

For the Invicid single fluid

$T_0^0=\frac{\partial\mathcal{L}}{\partial{}_a^cD_t^\alpha\varphi}{}_a^cD_t^\alpha\varphi-\mathcal{L}$(32)

$$T_0^0=-\frac{1}{2}\frac{({}_a^cD_t^\alpha\varphi)^2}{{}_a^cD_{x_i}^\alpha\varphi}-\mu{}_a^cD_{x_i}^\alpha\varphi$$

$T_i^0=\frac{\partial\mathcal{L}}{\partial{}_a^cD_t^\alpha\varphi}{}_a^cD_{x_i}^\alpha\varphi=-\frac{{}_a^cD_t^\alpha\varphi}{{}_a^cD_{x_i}^\alpha\varphi}{}_a^cD_{x_i}^\alpha\varphi$(32)

$$T_i^0=-{}_a^cD_t^\alpha\varphi$$

$$T_0^i = \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^\alpha \varphi}\, {}_a^cD_t^\alpha \varphi = \left(\frac{1}{2}\frac{({}_a^cD_t^\alpha \varphi)^2}{\left({}_a^cD_{x_i}^\alpha \varphi\right)^2} + \mu\right) {}_a^cD_t^\alpha \varphi \quad (33)$$

$$T_0^i = \frac{1}{2}\frac{\left({}_a^cD_t^\alpha \varphi\right)^3}{\left({}_a^cD_{x_i}^\alpha \varphi\right)^2} + \mu\, {}_a^cD_t^\alpha \varphi$$

$$T_j^i = \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^\alpha \varphi}\, {}_a^cD_{x_j}^\alpha \varphi = \left(\frac{1}{2}\frac{({}_a^cD_t^\alpha \varphi)^2}{\left({}_a^cD_{x_i}^\alpha \varphi\right)^2} + \mu\right) {}_a^cD_{x_j}^\alpha \varphi$$

(34)

$$T_i^i = \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^\alpha \varphi}\, {}_a^cD_{x_i}^\alpha \varphi - \mathcal{L} \quad (35)$$

$$T_i^i = \left(\frac{1}{2}\frac{({}_a^cD_t^\alpha \varphi)^2}{\left({}_a^cD_{x_i}^\alpha \varphi\right)^2} + \mu\right) {}_a^cD_{x_i}^\alpha \varphi + \frac{1}{2}\frac{({}_a^cD_t^\alpha \varphi)^2}{{}_a^cD_{x_i}^\alpha \varphi} - \mu\, {}_a^cD_{x_i}^\alpha \varphi$$

(36)

$$T_i^i = \left(\frac{1}{2}\frac{({}_a^cD_t^\alpha \varphi)^2}{{}_a^cD_{x_i}^\alpha \varphi} + \mu\, {}_a^cD_{x_i}^\alpha \varphi\right) + \frac{1}{2}\frac{({}_a^cD_t^\alpha \varphi)^2}{{}_a^cD_{x_i}^\alpha \varphi} - \mu\, {}_a^cD_{x_i}^\alpha \varphi \quad (37)$$

5. CONCLUSION

In this work we have shown that the Caputo's fractional derivative can be used to write the fractional form of the Lagrangian density and the Hamiltonian density for single fluid, then the fractional equations that described the motion of fluids can be obtained from the Euler-Lagrangian equation, and the energy-stress tensor in the fractional form were obtained also, after that we found, the equations of motion from. By after comparing the results of the fractional Euler-Lagrangian equations and the fractional Hamiltonian equations with those in classical results, we observed that the results in classical form are only a special case of the fractional form.

REFERENCES

[1] Debnath. L, (2003), Recent Applications of Fractional Calculus to Science and Engineering, Hindawi Publishing Corp.

[2] Bernard.f.Schutz , (1970) , Perfect Fluid in General Relativity : Velocity Potentials and a Variational Principle , Phys. Rev. D 2,2762.

[3] Saarloos.w.Van , Bedeaux.D , Mazur.P , (1980), Hydrodynamics For An Ideal Fluid: Hamiltonian Formalism And Liouville – Equation , Instituut-Lorentz, Rijksuniversiteitte Leiden, Nieuwsteeg 18, 2311 SB Leiden, The Netherlands.

[4] Ambrosi.D , (1998) , Hamiltonian formulation for surface waves in a layered fluid , ELSEVIER, Wave Motion 31(2000)71-76.

[5] Manoff.S , (2001), Lagrangian fluid mechanics , Bulgarian Academy of Sciences ,Institute for Nuclear Research and Nuclear Energy , Department of Theoretical Physics Blvd, TzarigradskoChaussee 72 , 1784 - Sofia, Bulgaria.

[6] Manoff.Sawa , Lagrangian Theory For Perfect Fluids , Bulgarian Academy of Sciences , Institute for Nuclear research and Nuclear Energy , Department of Theoretical Physics, Blvd. TzarigradskoChaussee 72,1784 Sofia-Bulgaria.

[7] Bhat.H.S , Fetecau.R.C , Marsden.J.E , Mohseni.K , West.M , (2003) , Lagrangian Averaging For Compressible Fluids , MULTISCALE MODEL. SIMUL. Vol. 3, No. 4, pp. 818-837, DOI. 10.1137/030601739. Society for Industrial and Applied Mathematics.

[8] Baleanu. D, Golmankhaneh. A. K, Baleanu. M. C, (2009), Hamiltonian Structure of Fractional First Order Lagrangian, Int J Theor Phys (2010) 49: 365-375, DOI 10. 1007/s10773-009-0209-5.

[9] Jaradat. E. K, Hijjawi. R. S, Khalifeh. J. M, (2010), Fractional Canonical Quantization of the Free Electromagnetic Lagrangian Density, Jordan journal of physics, volume3, number2, 2010.pp. 47-54.

[10] McDonough. J. M, Lectures in elementary fluid dynamics: Physics, Mathematics and Applications, Departments of Mechanical Engineering and Mathematics, University of Kentucky, Lexington, KY 40506-0503, ©1987. 1990. 2002. 2004. 2009.

[11] Clausse. A, Lagrangian Density Equations of Single-Fluid and Two-Fluid Flows, CNEA, CONICET and Universidad Nacional del Centro, 7000 Tandil, Argentina.

[12] Mandl. F, Shaw. G, Quantum Field Theory, department of theoretical physics, the Schuster laboratory,The University Manchester, AWiley-Interscience Publication, John Wiley& Sons, Chichester. New York. Brisbane. Toronto. Singapore.

[13] McMahon. D, (2008), Quantum Field Theory Demystified, The McGraw-Hill Companies.