APPLICATION OF HOMOTOPY ANALYSIS NATURAL TRANSFORM METHOD TO THE SOLUTION OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT: *A new Homotopy analysis natural transform method (HANTM) has been proposed in this article. The method has been suggested for the solution of various linear and nonlinear Fokker-Plank equations; it is a combine form of the Natural Transform method and the Homotopy Analysis method. The suggested techniques seek the solution without any restrictive assumptions; it also shown the round-off errors. The outcome of the study indicates that the approach is result-oriented, simple and applicable to other nonlinear mathematical problems.*

Keywords: Homotopoy analysis method; Natural transform; Fokker-Plank equations; Linear and nonlinear partial differential equations.

1 INTRODUCTION

Non-linear phenomena that is an integral part of many scientific fields, for example solids state physics, plasma physics, fluid mechanics, population dynamical models and chemical kinetics, can be modeled by nonlinear differential equations. In a lot of disciplines, related to science and engineering, it is considered significant to get to exact or numerical solution of the nonlinear partial differential equations. It is quite difficult to find an exact and numerical solution of nonlinear equations, it often requires new methods for finding the exact and approximate solutions. Certain mathematical methods- Adomian Decomposition Method (ADM) [1,6,7,39], Homotopy Perturbation Method (HPM) [3,6,7,11,14,18], Homotopy Analysis Method (HAM) [6,7,12,23,25,26,27,28,29,33], Variation Iteration Method (VIM) [6,7,15,16,17,19], Laplace Decomposition Method (LDM) [20,24,37], Homotopy Perturbation Sumudu Transform Method (HPSTM) [22] and Homotopy Analysis Transform Method (HATM) [21] have been proposed in the study to obtain exact and approximate analytical solutions of nonlinear partial differential and ordinary differential equations.

A new approximate method Homotopy Analysis Natural Transform Method (HANTM) has been introduced for the solution of nonlinear partial differential equations in the article under consideration . The proposed method is a combination of Natural transform method and Homotopy analysis method. Moreover, convergent series method is the main focus of the work, coupled with certain computable components in a direct way. This work does not take into consideration linearization, perturbation or restrictive assumptions. The combination of two powerful methods for getting the exact and approximate analytical solutions for nonlinear partial differential equations is the hall mark of this work. The article underscores the effectiveness of Homotopy Analysis Natural Transform Method (HANTM).

2 Preliminaries

Definition 2.1: Let $f(t)$ be a function defined for all $t \geq 0$ .Then Natural transform of $f(t)$ is the function $R(s,u)$ defined by

$$R(s,u) = \int_0^\infty f(ut)e^{-st}dt$$

provided that the integral on the right exists. Where $s,u \in (-\tau,\tau)$ while transform is defined over the set of function given by

$$A = \left\{ f(t) \middle| \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|\tau|}{\tau j}}, if t \in (-1) \times [0,\infty) \right\}$$

. (2.1)

3 Fokker-Planck equations

In order to explain the Brownian motion of particles, Fokker and Planck’s expounded the Fokker-Planck equation [31]. Natural sciences have been exploiting this equation in various fields; Quantum optics, Solid state physics, Chemical physics, and Theoretical biology and Circuit theory. Fokker-Planck equations underscores the erratic motions of tiny particles that are submerged; it also explains the fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows and the stochastic aspects of exchange rates. It could be inferred that Fokker-Planck equations are applicable on equilibrium and non- equilibrium systems [9,13,30,36]. The general form of Fokker-Planck equation is

$$\frac{dU}{dt} = \left[-\frac{d}{dx}A(x) + \frac{d^2}{dx^2}B(x) \right]U, \tag{3.2}$$

with the initial condition

$$U(x,o) = f(x), x \in R$$

.

$A(x)$ and $B(x)$ are describe as diffusion and drift coefficients, where in $U(x,t)$ is an unknown function .Both the diffusion and drift coefficients in equation (3.2) can be functions of x and t as well

$$\frac{dU}{dt}=\left[-\frac{d}{dx}A(x,t)+\frac{d^2}{dx^2}B(x,t)\right]U, \quad (3.3)$$

equation (3.2) is also named as a Forward Kolmogorov equation. Backward one is another type of this equation

$$\frac{dU}{dt}=\left[-A(x,t)\frac{d}{dx}+B(x,t)\frac{d^2}{dx^2}\right]U. \quad (3.4)$$

A generalization of equation (3.2) to N-variables of $x_1,x_2,....x_N$, yields to

$$\frac{dU}{dt}=\left[-\sum_{i=1}^{N}\frac{d}{dx_i}A_i(x)+\sum_{i,j=1}^{N}\frac{d^2}{dx_i dx_j}B_{i,j}(x)\right]U, \quad (3.5)$$

with the initial condition

$$U(x,0)=f(x), x=(x_1,x_2,.....,x_N)\in R^N.$$

A relatively more general form of linear equation is the non-linear Fokker-Planks equation; it is used in Plasma physics, Surface physics, Astrophysics, The physics of polymer fluids and particle beams, Nonlinear Hydrodynamics, Theory of electronic-circuitry and Laser arrays, Engineering, Biophysics, Population dynamics, Human movement sciences, Neurophysics, Psychology and Marketing [10]. The nonlinear form of the Fokker-Planck equation could be expressed in formulated form as:

$$\frac{dU}{dt}=\left[-\frac{d}{dx}A(x,t,U)+\frac{d^2}{dx^2}B(x,t,U)\right]U. \quad (3.6)$$

A generalization of equation (3.6) with N-variables $x_1,x_2,.......,x_N$, yields to

$$\frac{dU}{dt}=\left[-\sum_{i=1}^{N}\frac{d}{dx_i}A_i(x,t,U)+\sum_{i,j=1}^{N}\frac{d^2}{dx_i dx_j}B_{i,j}(x,t,U)\right]U. \quad (3.7)$$

4 The Fundamental idea of Homotopy Analysis method (HAM)

To exponent the basic idea of HAM, the following differential equation is given

$$N\left[V(x,t)\right]=0. \quad (4.8)$$

Here in N donates a nonlinear operator, x and t denotes the independent variables while V represents an unknown, unspecific function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the HAM , we first construct the so-called zeroth-order deformation equation as

$$(1-p)L\left[\varphi(x,t;p)-V_0(x,t)\right]=\hbar pH(x,t)N[V(x,t)]. \quad (4.9)$$

Where $p\in[0,1]$ is the embedding parameter, $h\neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\varphi(x,t,p)$ is an unknown function, $V_0(x,t)$ is an initial guess of $V(x,t)$ and $H(x,t)$ denotes a nonzero auxiliary function. Obviously, when the embedding parameter p= 0 and p= 1 gives respectively

$$\varphi(x,t;0)=V_0(x,t) \quad \text{and} \quad \varphi(x,t;1)=V(x,t), \quad (4.10)$$

The equation indicates that with the increase of p from 0 to L the solution $\varphi(x,t,p)$ undergoes from the initial guess $V_0(x,t)$ to $V(x,t)$, if we expand $\varphi(x,t,p)$ in Taylor series with respect p then

$$\varphi(x,t;0)=V_0(x,t)$$

$$\varphi(x,t;p)=V_0(x,t)+\sum_{n=1}^{\infty}V_n(x,t)p^n. \quad (4.11)$$

Where

$$V_n(x,t)=\frac{1}{n!}\frac{d^n\varphi(x,t;p)}{dp^n}\mid p=0. \quad (4.12)$$

If the five factor-the auxiliary linear operator ,the initial guess the auxiliary parameter h, and the auxiliary function-all are well selected, the series (4.11) converges at p = 1, then we have

$$V(x,t)=V_0(x,t)\sum_{n=1}^{\infty}V_n(x,t), \quad (4.13)$$

It is one of the solution beloning to original nonlinear equations. Equation(4.13) refers that the governing equation can be deduced from the zero-order deformation equation (4.9).

$$\vec{V}_n=\{V_0(x,t),V_1(x,t),....,V_n(x,t)\}, \quad (4.14)$$

differentiating the zeroth-order deformation equation (4.9) n-times with respect to p and then dividing them by n!and finally setting p = 0, we get the following nth-order deformation equation as

$$L[V_n(x,t)-X_nV_{n-1}(x,t)]=\hbar H(x,t)\Re_n(\vec{V}_{n-1}), \quad (4.15)$$

where

$$\Re_n(\vec{V}_{n-1}) = \frac{1}{(n-1)!}\frac{d^{n-1}N[\varphi(x,t;p)]}{dp^{n-1}}\mid p=0, \quad (4.16)$$

and

$$X_n=\begin{cases}0, n\leq 1,\\ 1, n>1.\end{cases} \quad (4.17)$$

5 Homotopy Analysis Natural Transform method (HANTM)

In order to describe the basic idea of the proposed method, an equation $N[U(x)]=g(x)$ is considered, where N donates a general nonlinear ordinary or partial differential operator

including both the linear and nonlinear terms. The former is decomposed into L+R, where L is the highest order linear operator and R is the remaining of the linear operator; hence the
equation can be denoted :

$$LU + RU + NU = g(x) \quad (5.18)$$

Where $N\ U$, indicates the nonlinear terms. By applying the Natural transform on both sides of equation (5.18), we get

$$N[L\ U] + N[R\ U] + N[N\ U] = N[g(x)]. \quad (5.19)$$

Using the differentiation property of the Natural transform, we have

$$\frac{N[U]}{u^n} - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} + N[RU] + N[NU] = N[g(x)] \quad (5.20)$$

On simplifying

$$N[U] - u^n \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(n-k)}} + \frac{u^n}{s^n}[N[RU] + N[NU] = N[g(x \quad (5.21)$$

The nonlinear operations is defined

$$N[\varphi(x,t;p)] = N[\varphi(x,t;p)] - \frac{u^n}{s^n} \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{s^{n-k}} \varphi^{(k)}(x,t;p) \quad (5.22)$$

where $p \in [0, 1]$ and $\varphi(x,t,p)$ is a real function of x, t and p. We construct a Homotopy as follows

$$(1-p)N[\varphi(x,t;p) - U_0(x,t)] = \mathrm{h}pH(x,t)N[U(x,t)] \quad (5.23)$$

'N" refers to the Natural transform, $p \in [0, 1]$ is the embedding parameter, $H(x, t)$ refers to a non-zero auxiliary function, $h \neq 0$ is an auxiliary parameter, $U_0(x,t)$ denotes initial guess of $U(x, t)$ and $\varphi(x,t,p)$ is a unknown function. Obviously, when the embedding parameter $p = 0$ and $p = 1$, it holds

$$\varphi(x,t;0) = U_0(x,t),$$
$$\varphi(x,t;1) = U(x,t) \quad (5.24)$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(x,t,p)$ undergoes variation from the initial guess to the solution $U_0(x,t)$. Expanding $\varphi(x,t,p)$ in Taylor series with respect to 'p', we have

$$\varphi(x,\ t;p) = U_0(x,\ t) + \sum_{m=1}^{\infty} U_m(x,\ t)p^m, \quad (5.25)$$

where

$$U_m(x,t) = \frac{1}{m!}\frac{d^m \varphi(x,t;p)}{dp^m}\Big|_{p=0} \quad (5.26)$$

The series (5.25) converges at p = 1, provided the auxiliary linear operator, the initial guess, the auxiliary h , and the auxiliary function are well-chosen; then

$$U(x,t) = U_o(x,t) + \sum_{m=1}^{\infty} U_m(x,\ t), \quad (5.27$$

It is one of the solutions related to the original nonlinear equations. The definition (5.27) refers to the governing equation that can be deduced from the zero-order deformation (5.23)

$$\vec{U}_m = \{U_o(x,t), U_1(x,t), ..., U_m(x,t)\}. \quad (5.28)$$

If the zero-order deformation equation (5.23) is differentiated m-times with respect to 'q' and then dividing them by *m*! Setting p = 0, the following mth-order deformation equation is the outcome:

$$N[U_m(x,t) - \chi_m U_{m-1}(x,t)] = \mathrm{h}H(x,t)\Re_m(\vec{U}_{m-1}). \quad (5.29)$$

Applying the inverse Natural transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t)] + \mathrm{h}N^{-1}[H(x,t)\Re_m(\vec{U}_{m-1})]. \quad (5.30)$$

$$\Re_m(\vec{U}_{m-1}) = \frac{1}{(m-1)!}\frac{d^{m-1}N[\varphi(x,t;p)]}{dp^{m-1}} \quad (5.31)$$

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1 \end{cases} \quad (5.32)$$

6 Applications of HANTM to Fokker-Planck equations

In order to solve linear and non linear Fokker-Planck equations the HANTM has been used in this section; hence the following linear Fokker-Planck equation is considerd:

Example 6.1. Consider the following linear Fokker-Planck equation

$$Ut = Ux + Uxx, \quad (6.33)$$

with the initial condition

$$U(x,0) = x. \quad (6.34)$$

According to the HANTM, we take the initial guess as

$$U_0(x,t) = x. \quad (6.35)$$

If method to (6.33) is applied, stringed with the initial conditions, the outcome will be

$$N[U] - x - \frac{u}{s}[N[Ux] + N[Uxx]] = 0 \quad (6.36)$$

The nonlinear operator is

$$N[\varphi(x,\ t;q)] = N[\varphi(x,\ t;q)] - x - \frac{u}{s}[N[\frac{d\varphi(x,t;p)}{dx}] + N[\frac{d^2\varphi(x,t;p)}{dx^2}]] \quad (6.37)$$

and thus

$$\Re m(\vec{U}_{m-1}) = N[U_{m-1}] - (1-\chi_m)x - \frac{u}{s}[N[\frac{dU_{m-1}}{dx}] + N[\frac{d^2U_{m-1}}{dx^2}]]. \quad (6.38)$$

The mth-order deformation equation is given by

$$N[U_m(x,\ t) - \chi_m U_{m-1}(x,\ t)] \ = \mathrm{h}\Re_m(\vec{U}_{m-1}) \quad (6.39)$$

Applying the inverse Natural transform, we have

$$U_m(x,\ t) = \chi_m U_{m-1}(x,\ t)] + \mathrm{h}N^{-1}[\Re_m(\vec{U}_{m-1})] \quad (6.40)$$

Solving above equation (6.40), for m = 1, 2, 3..., we get

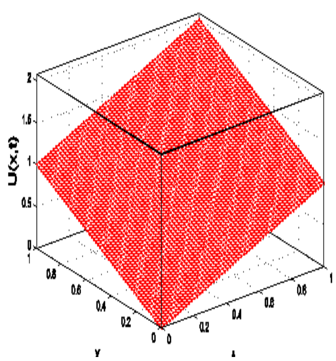
$$U_1(x,\ t) \ = -\mathrm{h}t,$$
$$U_2(x,\ t) \ = -\mathrm{h}(1 \ + \mathrm{h})t, \quad (6.41)$$
$$U_3(x,\ t) \ = -\mathrm{h}(1 \ + \mathrm{h})^2 t,$$

and so on.

Taking $\mathrm{h} = -1$, the solution is given by

$$U(x,\ t) \ = \sum_{m=0}^{\infty} U_m(x,\ t) \ = x + t, \quad (6.42)$$

which is the exact solution.



Plot of obtained exact solution of $U(x,t)$ of example 6.1.

The plot indicates that the solution that resulted from the underconsideration method is the same obtained by ADM [35], VIM [32] and HPM [8]. So the proposed method is in parallel with the exact solution ;moreover this method is the generalization of Homotopy Analysis Laplace and Homotopy Analysis Sumudu transform perturbations methods.

Example 6.2. Consider the following linear Fokker-Planck equation (3.3) such that

$$A(x,t) = e^t \coth x \cosh x + e^t \sinh x - \coth x$$
$$B(x,\ t) \ = e^t \cosh x$$
$$U_t = -\frac{d}{dx}A(x,t)U + \frac{d^2}{dx^2}B(x,t)U, \quad (6.43)$$

with the initial condition

$$U(x,0) \ = \ sinhx,\ x \in R. \quad (6.44)$$

According to the HANTM, we take the initial guess as

$$U_0(x,\ t) \ = \ sinhx. \quad (6.45)$$

In case the aforesaid method is applied ,provided to initial condition, we have the following outcome;

$$N[U] - \sinh x - \frac{u}{s}N\left[-\frac{d}{dx}A(x,t)U + \frac{d^2}{dx^2}B(x,t)U\right] = 0\ . \quad (6.46)$$

The nonlinear operator is

$$N[\varphi(x,\ t;p)] \ = N[\varphi(x,\ t;p)] - sinhx$$
$$-\frac{u}{s}N\left[-\frac{d}{dx}A(x,t)\varphi(x,t;p) + \frac{d^2}{dx^2}B(x,t)\varphi(x,t;p)\right]. \quad (6.47)$$

And thus

$$\Re m(\vec{U}_{m-1}) \ = N[U_{m-1}] - (1-\chi_m)\ sinhx$$
$$-\frac{u}{s}N\left[-\frac{d}{dx}A(x,t)U_{m-1} + \frac{d^2}{dx^2}B(x,t)U_{m-1}\right]. \quad (6.48)$$

The mth-order deformation equation is given by

$$N[U_m(x,t) - \chi_m U_{m-1}(x,t)] = \mathrm{h}\Re m(\vec{U}_{m-1}). \quad (6.49)$$

Applying the inverse Natural transform, we have

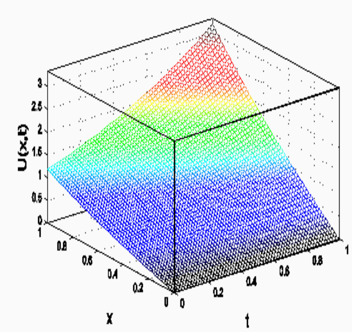
$$U_m(x,t) = \chi_m U_{m-1}(x,t)] = \mathrm{h}N^{-1}[\Re m(\vec{U}_{m-1})]. \quad (6.50)$$

Solving above equation (6.50), for m = 1, 2, 3..., we get

$$U_1(x,\ t) \ = -\mathrm{h}t \sinh x,$$
$$U_2(x,\ t) \ = -\mathrm{h}(1+\mathrm{h})t \sinh hx + \frac{\mathrm{h}^2 t^2 \sinh x}{2} \quad (6.51)$$
$$U_3(x,\ t) \ = -\mathrm{h}(1+\mathrm{h})^2 t \sinh hx + \mathrm{h}^2(1+\mathrm{h})t^2 \sinh x - \frac{\mathrm{h}^3 t^3 \sinh x}{6},$$

And so on taking $h \neq -1$, the solution is given by

$$U(x,\ t) \ = e^t sinhx, \quad (6.52)$$



Plot of obtained exact solution of $U(x,t)$ of example 6.2.

It is the exact solution. Both the plot and the computational results indicate that our solution is closely linked with exact solutions; besides the proposed method is same as obtain by ADM [35], VIM [32] and HPM [8].

Example 6.3. In case the Backward Kolmogorov equation (3.4) considered, we have

$$A(x,\ t) \ = -(x+\ 1),\ B(x,\ t) \ = x^2 e^t \quad (6.53)$$

i.e $U_t = \ (x+\ 1)U_x + x^2 e^t U_{xx},$ (6.54)

With initial condition

$$U(x,0) = x+1,\ x \in R. \qquad (6.55)$$

The HANTM leads us to the initial guess as

$$U_0(x,\ t) = (x+1). \qquad (6.56)$$

If the above-mentioned method, coupled with the initial condition, is applied we have

$$N[U]-(x+1)-\frac{u}{s}N[(x+1)U_x + x^2 etU_{xx}] = 0$$

(6.57)

The nonlinear operator is

$$N[\varphi(x,t;p)] = N[\varphi(x,t;p)]-(x+1)$$

$$-\frac{u}{s}N\left[(x+1)\frac{d\varphi(x,t;p)}{dx}+x^2e^t\frac{d^2\varphi\ (x,t;p)}{dx^2}\right]$$

(6.58)

And thus

$$\Re m(\vec{U}_{m-1}) = N[U_{m-1}]-(1-\chi_m)(x+1)$$

$$-\frac{u}{s}N\left[(x+1)\frac{dU_{m-1}}{dx}+x^2e^t\frac{d^2\ U_{m-1}}{dx^2}\right] \qquad (6.59)$$

The mth-order deformation equation is denoted by

$$N[U_m(x,t)-\chi_m U_{m-1}(x,t)] = \mathrm{h}\Re_m(\vec{U}_{m-1}). \qquad (6.60)$$

In case the inverse Natural transform is applied we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t)] + \mathrm{h}N^{-1}[\Re_m(\vec{U}_{m-1})]. \qquad (6.61)$$

Solving above equation (6.61), for m = 1, 2, 3..., we get

$$U_1(x,t) = -\mathrm{h}(x+1)t,$$

$$U_2(x,t) = -\mathrm{h}(1+\mathrm{h})\ (x+1)t+\frac{\mathrm{h}^2(x+1)t^2}{2}$$

(6.62)

$$U_3(x,t) = -\mathrm{h}(1+\mathrm{h})^2\ (x+1)t+\frac{\mathrm{h}^2(1+\mathrm{h})(x+1)t^2}{2}-\frac{\mathrm{h}^3(x+1)t^3}{6}$$

and so on.

If $h \neq -1$, the solution could be denoted by;

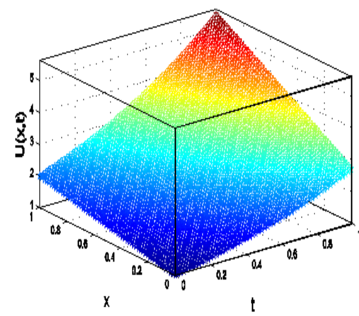
$$U(x,t) = e^t(x+1), \qquad (6.63)$$

It is the exact solution and is in parallel with that obtained by ADM [35], VIM [32] and HPM [8]. Plot of exact solution is given in figure 6.3, which is similar to that of exact solution obtained in [32], [35] and in [8] etc.

7 CONCLUSION

The Homotopy Analysis Natural transform method (HANTM)has been introduced in this work in order to solve non-linear equations. How it is workable,practicable and efficient ,it has been applied for the solutions of linear and nonlinear Fokker-Planck equations. The outcome obtained using the HANTM are in complete parallel with the outcome reached through ADM [35], VIM [32] and HPM [8]. The proposed technique could reduce the volume of the computational work as compared to the classical methods inspite of the fact that the high accuracy of the numerical result will maintain; the size reduction and improvement of the performance approach are inter-related

Eventually, it was conclude that the HANTM is an effective and powerful method to find analytical and numerical solutions for various linear and nonlinear partial differential equations.



Plot of obtained exact solution of $U(x,t)$ of example 6.3.

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