ESTIMATION OF MAXIMUM RETURNS OF SUGARCANE PRODUCTION QUADRATIC PROGRAMMING MODEL USING GRADIENT AND CONVEX APPROACH

S. Mustafa[1], R. Shah[2], S. Joshi[3]
[1,2]Department of Mathematics and Statistics, PMAS, Arid Agriculture University Rawalpindi
[1] saimamustafa28@gmail.com,
[2.] muhammadrafayshah@gmail.com
[3] Department of Mathematics, Walchand College of Engineering, India
[3]santosh.joshi@walchandsangli.ac.in

ABSTRACT: *In this paper, quadratic programming model subject to linear constraint is used in order to obtain the best optimum values of the cost parameters under certain limitations to attain maximum net returns. To maximize the quadratic function subject to the linear constraints, calibration constant is implemented in the constraints. This made the model distinguishable from the usual LP format. For finding maxima and minima convex analysis approach along with Lagrange multipliers is analyzed. These techniques are implemented for estimating unknown parameters. Hence the obtained estimated values of cost parameters illustrate that the model attains the maximum net returns. The data on dependent and explanatory variables are collected from different secondary sources to originate the models. Then the estimated values of above mentioned cost parameters are applied on the time series data of sugarcane. The results obtained from the data produce a unique optimal solution.*

Keywords: Quadratic programming;, Convexity, Calibration, Lagrange's Multipliers

INTRODUCTION:

Mathematical optimization relates to the assortment of a paramount element from some set of accessible alternatives. Greek mathematicians sorted out several optimization problems that were related to geometric studies. To ascertain optimization Fermat and Lagrange establish the formulas which depend on calculus. Newton and Gauss anticipated iterative schemes for moving towards a best possible solution. If we look at the pages of history linear programming was the preliminary phrase for optimization [1]. Optimization means to resolve problems in which one search to capitalize or to diminish a real function by electing the values of real or integer variables from a set of limitations. In simple words we can say, that, optimization stands for discovering "best available solution" of some objective function given by a set of constraints. Optimization has wide applications in statistics, engineering and economics.

Here we will deal; Non-linear programming which is generalization of linear programming and especially well studied non-linear programming is quadratic programming. Quadratic Programming (QP) model is defined as follows

In quadratic programming, the objective function is quadratic subject to linear constraints.

Mathematically we define as

Maximize $f(x)= cx - 1/2x^tQx$ subject to $Ax \leq b$, and $x \geq 0$, where c is row vector , x and b are column vectors, Q and A are matrices and the superscript T denotes the transpose subject to linear constraint. In order to get the unique optimal solution, many algorithms for solving QP problems have been developed under the additional assumption that the objective function is convex.

Here we have used the convexity assumption in the objective function and linear constraints. The convexity approach has a greater importance in optimization problems. A unique category of optimization problems identified as convex optimization problem can proficiently used to sort out the solution in many cases. It has wide applications in different areas such as in statistics and economics. Scientists and practitioners from various fields and backgrounds are practicing the convex optimization technique in various tasks and researches. Bertsimes *et al.* [2] proposed a convex optimization approach to solving the nonparametric regression estimation when the underlying function is Lipschitz continuous. Tao et.al [3] has also studied convex analysis approach to difference of convex functions programming and gave the results about difference of convex functions duality, local and global optimalities in difference of convex functions programming. Exact penalty, Lagrangian duality without gap, and regularization techniques were evaluated. Binmore and Davies, introduced the Hessian approach for convexity.

MATERIAL AND METHODS

In order to get the unique optimal solution we have used the convex approach along with gradient based technique named as Lagrange multiplier which is one of the most constructive techniques for finding such maxima and minima. The method of Lagrange multipliers can also accommodate multiple constraints. Apart from this we have applied calibration on the constraints. Calibration is a progression for formulating the recital of parameters of a system by contrasting it with précised values. Howitt [4] , used calibration in agricultural production models. Goal of our research is to maximize the net returns to management, in the first stage for achieving the goal a quadratic function is developed subject to the linear constraint. The set of calibration constraint distinguish the model from the ordinary LP layout. To precisely estimate the capability of the model, assumptions of convexity and concavity are checked with the help of Hessian matrix. The objective function is then transformed into Lagrangian function. The major aim of the study is to estimate the cost parameters. The model is again reformulated and some adjustments are made. The reformulation is done by the addition of one more constraint in the model. The above mentioned adjustment is the addition of observed production cost. Evaluated values of cost parameters are applied on the above mentioned time series data of sugarcane to express the best optimal solution. The results obtained from the data produce a unique optimal solution.Using the above concepts, we have the following results.

Data Analysis

Consider

$$Z = \left.\begin{array}{l} \max \sum_i x_i y_i c_i - (\beta_i + \frac{1}{2}\varphi_i c_i) c_i \\ \text{s.t} \\ A_i c_i \le b_i \ , \end{array}\right\} i = 1, 2, \ldots\ldots, n \qquad (1)$$

Whre x_i is the production of the sugarcane, y*i* is yield tons per hectare, c_i is the cultivated area of sugarcane, A_i is the cropped area of sugarcane, b_i is the total cropped area of Pakistan, whereas (β_i, ϕ_i) are cost parameters. In order to maximize the objective function, we will proceed in the following way. In the first stage, we will take first and second order partial derivatives of the objective function and insert these results in Hessian matrix of order (3 x 3). The second order conditions are verified by constructing the Bordered Hessian and largest principal minor. Since first and second order principal minors are less than or equal to zero, and the principal minors are alternate in sign with the determinant of the Bordered Hessian itself, therefore, the given function is negative semi definite which is concave and the solution is a local maximum. This condition guarantees that there exists the unique optimal solution of the function. The Lagrangian function for the problem may be written as:

$$Z = \sum_i (x_i y_i c_i - \beta_i c_i - \frac{1}{2}\varphi_i c_i^2) + \tau(b_i - A_i c_i)$$

Here τ is the Lagrange multiplier on the calibration constraint. The first order necessary conditions to the problem may be written

$$Z_c = x_i y_i - \beta_i - \varphi_i c_i - A_i \tau_i = 0$$

$$Z_\tau = b_i c_i - A_i c_i = 0$$

These conditions may be solved for the optimal solutions, c_i^* which in turn can be used to compute crop-specific profits using the formula:

$$\pi_i^* = x_i y_i c_i^* - (\beta_i + \frac{1}{2}\varphi_i c_i^*) c_i^*$$

$$\pi_i^* = x_i y_i c_i^* - (\beta_i + \frac{1}{2}\varphi_i c_i^*) c_i^* \qquad (2)$$

Estimates of the values in expression (2) can be computed from observed data. In particular, the researcher does observe the optimal solutions, c_i^* as well as the profit earned by the average producer, π_i^*. The latter can be determined from observed production, yield, and production costs. The first step in determining the cost parameters is to solve the following problem. Similar to expression (1) and letting $\kappa_i = \beta_i + \phi_i c_i$, where κ_i denote the observed production costs on crop i.

We have

$$\left.\begin{array}{l} \max \sum_i x_i y_i c_i - \kappa_i c_i \\ \text{s.t.} \ \ A_i c_i \le b_i \\ \qquad c_i \ \le \ c_i^* + \nu \end{array}\right\} \qquad (3)$$

Here the new constraint is also a calibration constraint, and ν is a small positive number known as a calibration constant. The Lagrangian for equation (3) can be written:

$$Z = \sum_i (x_i y_i c_i - \kappa_i c_i) + \tau(b_i - A_i c_i) + \eta(c_i^* + \nu - c_i)$$

Similarly η_i^* is the Lagrange multiplier on the calibration constraint. The first order necessary conditions to the calibration problem are:

$$Z_c = x_i y_i - \kappa_i - A_i \tau_i - \eta = 0 \qquad (4)$$

$$Z_\tau = b_i - A_i c_i = 0 \qquad (5)$$

$$Z_\eta = c_i^* + \nu_i - c_i = 0 \qquad (6)$$

By applying Gauss Jordan method results of equation (4) to equation (5), we have,

$$c^* = \frac{b_i}{A_i}$$

$$\tau^* = \frac{x_i y_i - \beta_i - \varphi_i c_i}{A_i}$$

$$\eta^* = \beta_i + \varphi_i c_i - \kappa_i$$

Now all parameters are known. By construction, its solutions will be within a small tolerance of the observed acreages c_i^* ; in the following discussion we have used c_i^* to denote both the observe acreage levels and the solutions to (3). Similarly, π_i^* will denote both the observed profits and those computed from the solutions to (3)

$$\pi_i^* = x_i y_i c_i^* - \kappa_i c_i^* \qquad (7)$$

Our calibration problem is one of obtaining values for (β_i, ϕ_i), such that if these values were inserted in problem (4.1) and it were solved numerically, the optimal solutions would equal c_i^* and the computed profits by crop would equal π_i^*. The information obtained from solving problem (4.1), namely the values of η_i^*, is needed for this calibration process. In Lagragian function if (β_i, ϕ_i) are set

Table 4.1 Sugarcane production, area, yield, supported price, nitrogen price and lagged value of production in Pakistan-1989-90 to 2004-05

Year	Production (M/T)	Yield per hectare (M/T)	Cultivated Area of Sugarcane (Million Acres)	Cropped Area of Sugarcane (Million Acres)	Total Cropped Area	Supported price (Rs per 40 kg)	Nitrogen price (Rs per 50 kg)	Lagged value
1989-90	35.49	41.56	2.11	51.74	53.03	13.75	185.00	36.98
1990-91	35.99	40.71	2.18	51.79	53.92	15.25	195.00	35.49
1991-92	38.87	43.38	2.21	52.04	53.67	17.75	195.00	35.99
1992-93	38.06	43.02	2.19	52.88	55.45	17.50	205.00	38.87
1993-94	44.43	46.14	2.38	53.15	54.04	18.25	210.00	38.06
1994-95	47.17	46.75	2.49	53.25	54.71	20.75	235.00	44.43
1995-96	45.23	46.96	2.38	53.57	55.82	21.75	267.00	47.17
1996-97	42.00	43.54	2.38	54.31	56.17	24.50	340.00	45.23
1997-98	53.10	50.28	2.61	54.27	56.93	36.00	344.00	42.00
1998-99	55.19	47.78	2.85	54.19	56.49	36.00	346.00	53.10
1999-00	46.33	45.88	2.50	54.27	56.19	36.00	327.00	55.19
2000-01	43.61	45.39	2.37	54.69	54.46	36.00	363.00	46.33
2001-02	48.04	48.06	2.47	55.03	54.66	43.00	394.00	43.61
2002-03	52.06	47.34	2.72	54.88	53.99	43.00	411.00	48.04
2003-04	53.42	49.74	2.65	54.66	56.69	40.00	421.00	52.06
2004-05	47.24	48.89	2.39	54.73	55.62	43.00	468.00	53.42

at their correct values then $x_i y_i - A_i \tau_i = \beta_i + \phi_i c_i$. By equation (4), we also know that $x_i y_i - A_i \tau_i = \kappa_i + \eta_i$. Combining these two relationships, we have

$$\beta_i + \varphi_i c_i = \kappa_i + \eta_i \tag{8}$$

By equations (4) and (6), if (β_i, ϕ_i) are set at their correct values then

$$x_i y_i c_i^* - (\beta_i + \frac{1}{2}\phi_i c_i^*) c_i^* = x_i y_i c_i^* - \kappa_i c_i^*$$

this equation reduces to

$$\kappa_i = \beta_i + \frac{1}{2}\phi_i c_i^* \tag{9}$$

Equations (8) and (9) are the system of two equations which uniquely determine the two (β_i, ϕ_i), given the observed data (c_i^*, κ_i) and the computed multiplier η_i^*. This system is solved explicitly as follows. $\frac{1}{2}\phi_i c_i^* = \eta_i$. Solving for ϕ_i, we have $\phi_i = {2\eta_i}/{c_i^*}$

Substituting the above values yields, $\beta_i + \eta_i = \kappa_i$, which can be solved for β_i as $\beta_i = \kappa_i - \eta_i$.

The main purpose of the problem under study is to estimate the values of ϕ_i and β_i which has been estimated. The above results have been implemented on the data given below which are also satisfying the above stated relations.

PRESENTATION AND ANALYSIS OF DATA

The cultivated area of sugarcane, production and yield per hectare for the last sixteen years from 1989-90 to 2004-05 are given in the Table 4.1. It is clear from Table 4.1 that the maximum values of cultivated area of sugarcane; yield and production is attained in the year 1997-98. The values of production and cultivated area of sugarcane are slightly less than the maximum value obtained in the year 1998-1999. But the difference is ignored because of the fact that yield gives maximum value in this mentioned year. For obtaining the values of cost parameters two values of Lagrange multiplier i.e. before and after using the calibration are applied separately on the data given in Table 4.1. When the value of Lagrange multiplier, before calibration is applied on the data we obtained the values mentioned in Table 4.2 denoted by τ^*. The values in this table show that maximum value of τ^* is obtained in the year 1997-98 which is 3.47. After using the calibration η^* represent the values of Lagrange multiplier which are also given in Table 4.2. The values in this table demonstrate that maximum value of η^* =4.32 which is also achieved in the year 1997-98. The comparison of the maximum value of τ^* obtained before the use of calibration with the maximum value of η^* attained after the use of calibration illustrates that after the use of calibration greater value is achieved. With the help of Lagrange multiplier the values of cost parameters are estimated. The estimated values of cost parameters β_i and ϕ_i along with the values of net returns are mentioned in Table 4.3. Table 4.3 demonstrates that the maximum net returns corresponding to the values of cost parameters is achieved in the year 1997-98. As explained above the maximum values of Lagrange multiplier as well as net returns is obtained in the same year 1997-98, therefore, a unique solution is achieved in this year.

Table4.2: Estimated values of area and Lagrange multipliers.

Year	κ	c*	τ^*	η^*
1989-90	1479.60	1.02	-4.27	-4.40
1990-91	1470.43	1.04	-5.00	-5.25
1991-92	1688.39	1.03	-2.41	-2.58
1992-93	1640.35	1.05	-2.82	-2.94
1993-94	2049.77	1.02	0.16	0.18
1994-95	2204.18	1.03	0.65	0.79
1995-96	2123.13	1.04	0.88	1.00
1996-97	1831.18	1.03	-2.10	-2.42
1997-98	2665.70	1.05	3.47	4.32
1998-99	2635.21	1.04	1.33	1.82
1999-00	2125.97	1.04	-0.06	-0.08
2000-01	1979.63	1.00	-0.48	-0.58
2001-02	2306.61	0.99	1.68	2.10
2002-03	2463.00	0.98	1.00	1.38
2003-04	2653.18	1.04	2.95	3.78
2004-05	2306.69	1.02	2.49	2.93

Table 4.3: Estimates of Net returns

Year	β_i	φ_i	Net Returns
1989-90	1484.00	-4.17	2949.92
1990-91	1475.68	-4.80	2929.40
1991-92	1690.98	-2.33	3371.06
1992-93	1643.29	-2.69	3274.28
1993-94	2049.59	0.15	4099.97
1994-95	2203.39	0.63	4410.31
1995-96	2122.13	0.84	4248.65
1996-97	1833.59	-2.03	3656.59
1997-98	2661.38	3.31	5342.66
1998-99	2633.39	1.28	5275.61
1999-00	2126.05	-0.06	4251.76
2000-01	1980.21	-0.48	3957.90
2001-02	2304.51	1.70	4618.40
2002-03	2461.62	1.02	4929.76
2003-04	2649.40	2.85	5316.38
2004-05	2303.76	2.45	4620.37

CONCLUSION

This research was mainly concerned with the estimation of cost parameters and maximization of net returns to management subject to the linear constraint. We have used convex quadratic programming (QP) model subject to linear constraint. To maximize the quadratic function subject to the linear constraints, calibration constant was implemented in the constraints, which makes the model distinguishable from the usual LP format. Assumptions of convexity and concavity were then verified by Hessian matrix for evaluating the ability of model accurately. For finding maxima and minima convex analysis approach along with Lagrange multipliers were analyzed. The results of unknown parameters were significant and provide a unique optimal solution when convex optimization techniques along with Lagrange's multipliers were implemented. From this study we can conclude that one can gain the unique and best optimal solution by using the convex optimization approach in quadratic programming model. For practical application secondary data for sugarcane was obtained. The data on dependent and explanatory variables were collected from different secondary sources to originate the models. With the help of Lagrange multiplier the values of cost parameters were estimated along with the net returns. Results of the study demonstrated that the maximum net returns corresponding to the values of cost parameters was achieved in the year 1997-98. By compiling all the results, maximum value of Lagrange multiplier as well as net returns was obtained in the same year 1997-98; therefore, a unique solution was achieved in this year.

REFERENCES

1. Diewert, W. E. 2008. Cost functions. The New Palgrave Dictionary of Economics. 2nd Edition.
1. 2. Bertsimas, D., D. Gamarnik and J. N. Tsitsiklis. 1999. Estimation of Time- Varying Parameters in Statistical Models; An Optimization Approach, J. Mach. Learning, 35(3): 225-245.
2. Tao, P. D. and L. T. H. An. 1997. Convex analysis approach to D. C. programming: theory, algorithms and applications. J. Acta Mathematica Vietnamica, 22(1): pp. 289.
3. Howitt, R. E. 1995. A calibration method for agricultural economic production models. J. Agricultural Economics, 46(2): 147-159.
4. Binmore and Davies. 2007. Calculus Concepts and Methods, Cambridge University Press:190 pp.
5. Goffe, W. L., G. D. Ferrier and J. Rogers. 1993. Global optimization of statistical mass functions with simulated annealing. J. Mathematical Programming. 46:1-40.
6. Sher, F. and E. Ahmed. 2008. Forecasting wheat production in Pakistan. J. Economics, 13(1) 57-85.