# EFFECT OF THE DIFFERENT FACTORS ON THE RELAXATION PARAMETER OF SOR GAUSS-SEIDEL METHOD

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***ABSTRACT****: In this paper a 2D Poisson's equation with Dirichlet boundary conditions is discretized by using the finite difference method. The discretized form of the equation leads to a system of linear equations of the form AX=B which is solved by Successive Over Relaxation (SOR) Gauss-Seidel (GS) method. The fast convergence of SOR GS method depends on the proper choice of the relaxation parameter (usually denoted by* $\omega$*) whose values typically lie in the range of (0, 2). But, in general, it is very difficult to find the best possible value of* $\omega$ *since there is no universal rule for all types of linear systems. However, for the linear systems obtained from the discretization of the Poisson's equation, different relations for the optimum value of* $\omega$ *have been proposed in the literature. Such relations usually depend on the step size or eigenvalues of A. However, in practice, there are many factors such as the boundary conditions, structure of coefficient matrix A and the column vector B, the forcing function, the error tolerance etc., which effect the* $\omega$ *and convergence behavior of SOR GS method. Therefore, this research is aimed at the analysis of the effect of the different factors on relaxation parameter of SOR GS method. To achieve this goal, the convergence behavior of solution is examined by varying the values of increment dx along x-axis, increment dy along y-axis; the left, right, top and bottom boundary conditions; the forcing function f, the length L of domain, the width W of domain and the error tolerance tol. The overall parametric analysis reveals that the feasible values of* $\omega$ *fall within the interval [0.4 1.8].*

**Keywords:** Poisson's equation, finite difference method, iterative methods, SOR Gauss-Seidel method, relation parameter, optimization of SOR parameter.

## INTRODUCTION

The systems of linear algebraic equations arise from the modeling of many scientific and engineering problems. In practice, the linear systems consisting of few variables are simple to solve by analytical methods but in case of the large number of unknowns the numerical methods are more efficient than analytical methods. Numerical methods work iteratively by starting from initial guess and update the solution in each step until the predefined error tolerance is achieved. A comprehensive overview of different iterative methods used to solve the linear systems of algebraic equations and their theory can be referred in [1-2]. The SOR Gauss Seidel GS method is a classical iterative method developed by [3] for the purpose of solving linear systems automatically on digital computers. The speed of convergence of SOR GS depends on the relaxation parameter $\omega$ but it is very difficult rather impossible to compute its best possible value in advance. If $\omega = 1$ the SOR GS method simplifies to the conventional Gauss Seidel method. A theorem stated by [4] shows that SOR GS fails to converge if $\omega$ is chosen outside the interval $(0, 2)$. Frequently, some heuristic estimate is used, such as $\omega = 2 - O(h)$ where $h$ is the mesh spacing of the discretization of the underlying physical domain [5]. For some specific model problems such as Poisson's equation the following formula gives the optimum value of $\omega$,

$$\omega_{opt} = \frac{2}{1+\sin(\pi h)}, \qquad (1)$$

where h = 1/(N + 1) is the mesh spacing. The exact formulation of Eq. (1) is demonstrated by [6]. Another approximation such as given below,

$$\omega_{opt} = 2(1-\pi h) + O(h^2), \qquad (2)$$

is worked out by [7] that involve the Taylor expansion of sine function. A comprehensive study was done by [8] to demonstrate and test the feasibility of applying Eq. (1) and Eq. (2) on 1D, 2D, 3D and d-D Poisson's equation discretized by finite difference method. They proved some fascinating results for the model problem and attempted to find the optimum values of relaxation parameter.

It was also shown by [9] that if $\omega_1$ and $\omega_2$ are chosen such that $\omega = \omega_1 + \omega_2 - \omega_1\omega_2$. Another relation was proposed by [10] by using the idea of lattice points that is the over-relaxation parameter $\omega$ can be tuned to optimize the convergence if

$$\omega \approx \frac{2}{1+\frac{\pi}{L}} \qquad (3)$$

Where L is the number of lattice points in the x or y directions.

For a 2-cyclic, consistently ordered, real symmetric, positive definite matrix A [11] pointed out that finding the optimum over-relaxation parameter $\omega$ is an important and often a difficult part of the problem. They suggested that only the spectral radius $\rho(\mathrm{A})$ of the associated Gauss-Seidel iteration matrix $\mathrm{A}$ is needed to determine the optimum parameter $\omega$. When $\rho(\mathrm{A})$ is close to unity, small changes in the estimate for $\rho(\mathrm{A})$ can drastically effect the rate of convergence of the SOR iterations and, thus an accurate estimate of $\rho(\alpha_1)$ is needed. [12] proved an optimal result for SOR applied to matrices with so-called property A, consistent ordering, and positive real eigenvalues of $D^{-1}A$. Given an maximal eigenvalue $\mu_{\max}$ of the undamped Jacobi iteration matrix $I - D^{-1}A$ ( $\mu_{\max} < 1$ is guaranteed by the assumptions in this case), the optimal damping factor for SOR is

$$\omega_{opt} = 1 + \left( \frac{\mu_{\max}}{1 + \sqrt{1 - \mu_{\max}^{2}}} \right)^2 \quad (4)$$

Which result a convergence rate of $\rho_{opt} = \omega_{opt} - 1$. Note that $\omega_{opt}$ approaches 2 when $\mu_{\max} \to 1$.

A similar relation for $\omega$ was worked out by [13] for a symmetric and positive definite matrix A. If it is known that all the eigenvalues $\lambda$ of A are real, and lies in the symmetric interval $-\mu \le \lambda \le +\mu < 1$, where $\mu$ is the spectral norm of A then he best choice of $\omega$ is given by

$$\omega = 1 + \left[ \frac{\mu}{1 + \left(1 - \mu^2\right)^{\frac{1}{2}}} \right]^2 \quad (5)$$

Many more optimum relations for $\omega$ can be found in literature with more complexity for example [3, 14-15]. However, most of them depend only on mesh spacing, eigenvalues or lattice points and do not include the effect of other important factors involved in the model. Therefore, this paper is devoted to evaluate the effect of the mesh spacing parameters dx along x-axis and dy along y-axis; the left, right, top and bottom boundary conditions; the forcing function f, the length L of domain, the width W of domain and the error tolerance tol on the SOR parameter.

## METHODOLOGY:

In order to analyze the effect of the different parameters on the SOR GS method; a 2D Poisson's equation with Dirichlet boundary conditions is described as model problem as follows:

$$-\Delta u = f, \quad (6)$$

or for 2D case:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f(x, y). \quad (7)$$

The geometric dimensions and boundary conditions are shown more clearly in the following figure.

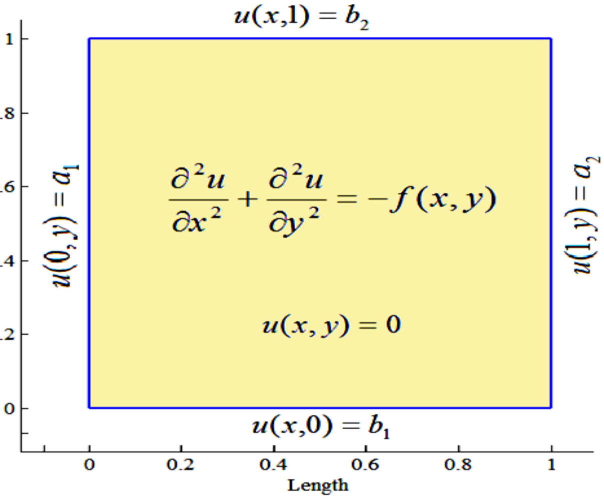


**Figure 1. Poisson's equation applied on a unit square with boundary conditions**

The output variable u may relate to any physical phenomenon but in this study the distribution of steady state temperature in a rectangular metallic plate is considered. The next step is to discretize Eq. (7) by using the central finite difference method as given below:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = -f(i,j) \quad , (8)$$

where h and k are the increment size; i and j are the indices of discrete mesh points along x and y axes respectively. Figure (2) exhibits a typical mesh with boundary conditions applied on all four sides of the plate.

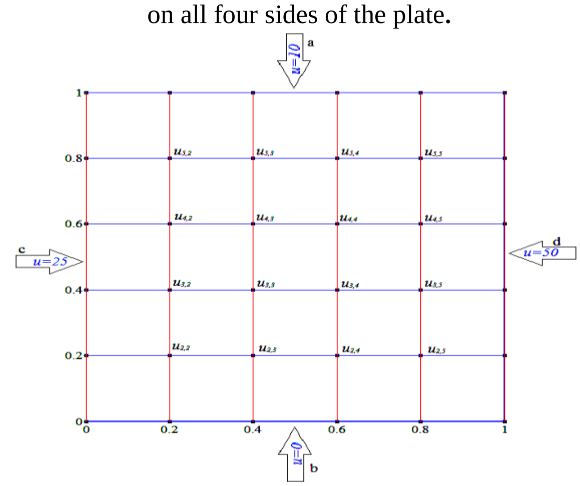


**Figure 2. Discretized domain indicating 5x5 mesh with 4x4 interior unknowns**

In particular case as shown in Figure (3) when the indices i and j are set as i=1, 2, 3, 4, 5 and j=1, 2, 3, 4, 5 in Eq. (8) the following system of linear equations is obtained:

$$
\left.\begin{aligned}
h^2 u_{3,2} - (2h^2 + 2k^2) u_{2,2} + h^2 b + k^2 u_{2,3} + k^2 c &= -f_{2,2} h^2 k^2 \\
h^2 u_{3,3} - (2h^2 + 2k^2) u_{2,3} + h^2 b + k^2 u_{2,4} + k^2 u_{2,2} &= -f_{2,3} h^2 k^2 \\
h^2 u_{3,4} - (2h^2 + 2k^2) u_{2,4} + h^2 b + k^2 u_{2,5} + k^2 u_{2,3} &= -f_{2,4} h^2 k^2 \\
h^2 u_{3,5} - (2h^2 + 2k^2) u_{2,5} + h^2 b + k^2 d + k^2 u_{2,4} &= -f_{2,5} h^2 k^2 \\
h^2 u_{4,2} - (2h^2 + 2k^2) u_{3,2} + h^2 u_{2,2} + k^2 u_{3,3} + k^2 c &= -f_{3,2} h^2 k^2 \\
h^2 u_{4,3} - (2h^2 + 2k^2) u_{3,3} + h^2 u_{2,3} + k^2 u_{3,4} + k^2 u_{3,2} &= -f_{3,3} h^2 k^2 \\
h^2 u_{4,4} - (2h^2 + 2k^2) u_{3,4} + h^2 u_{2,4} + k^2 u_{3,5} + k^2 u_{3,3} &= -f_{3,4} h^2 k^2 \\
h^2 u_{4,5} - (2h^2 + 2k^2) u_{3,5} + h^2 u_{2,5} + k^2 d + k^2 u_{3,4} &= -f_{3,5} h^2 k^2 \\
h^2 u_{5,2} - (2h^2 + 2k^2) u_{4,2} + h^2 u_{3,2} + k^2 u_{4,3} + k^2 c &= -f_{4,2} h^2 k^2 \\
h^2 u_{5,3} - (2h^2 + 2k^2) u_{4,3} + h^2 u_{3,3} + k^2 u_{4,4} + k^2 u_{4,2} &= -f_{4,3} h^2 k^2 \\
h^2 u_{5,4} - (2h^2 + 2k^2) u_{4,4} + h^2 u_{3,4} + k^2 u_{4,5} + k^2 u_{4,3} &= -f_{4,4} h^2 k^2 \\
h^2 u_{5,5} - (2h^2 + 2k^2) u_{4,5} + h^2 u_{3,5} + k^2 d + k^2 u_{4,4} &= -f_{4,5} h^2 k^2 \\
h^2 a - (2h^2 + 2k^2) u_{5,2} + h^2 u_{4,2} + k^2 u_{5,3} + k^2 c &= -f_{5,2} h^2 k^2 \\
h^2 a - (2h^2 + 2k^2) u_{5,3} + h^2 u_{4,3} + k^2 u_{5,4} + k^2 u_{5,2} &= -f_{5,3} h^2 k^2 \\
h^2 a - (2h^2 + 2k^2) u_{5,4} + h^2 u_{4,4} + k^2 u_{5,5} + k^2 u_{5,3} &= -f_{5,4} h^2 k^2 \\
h^2 a - (2h^2 + 2k^2) u_{5,5} + h^2 u_{4,5} + k^2 d + k^2 u_{5,4} &= -f_{5,5} h^2 k^2
\end{aligned}\right\} \quad (9).
$$

Where $a,b,c$ and $d$ are the Dirichlet boundary conditions. Equation (9) is further simplified to obtain the system matrix A, the unknown values of dependent variable u and the column vector B. The linear system is represented in standard matrix equation $AU = B$ as shown by the following Eq. (10):

$$
\begin{bmatrix}
-\alpha & k^2 & 0 & 0 & h^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k^2 & -\alpha & k^2 & 0 & 0 & h^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & k^2 & -\alpha & k^2 & 0 & 0 & h^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k^2 & -\alpha & 0 & 0 & 0 & h^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h^2 & 0 & 0 & 0 & -\alpha & k^2 & 0 & 0 & h^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & h^2 & 0 & 0 & k^2 & -\alpha & k^2 & 0 & 0 & h^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & h^2 & 0 & 0 & k^2 & -\alpha & k^2 & 0 & 0 & h^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & h^2 & 0 & 0 & k^2 & -\alpha & 0 & 0 & 0 & h^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h^2 & 0 & 0 & 0 & -\alpha & k^2 & 0 & 0 & h^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & h^2 & 0 & 0 & k^2 & -\alpha & k^2 & 0 & 0 & h^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & h^2 & 0 & 0 & k^2 & -\alpha & k^2 & 0 & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h^2 & 0 & 0 & k^2 & -\alpha & k^2 & 0 & 0 & h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h^2 & 0 & 0 & 0 & -\alpha & k^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h^2 & 0 & 0 & k^2 & -\alpha & k^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h^2 & 0 & 0 & k^2 & -\alpha & k^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h^2 & 0 & 0 & k^2 & -\alpha
\end{bmatrix}
\begin{bmatrix}
u_{2,2} \\ u_{2,3} \\ u_{2,4} \\ u_{2,5} \\ u_{3,2} \\ u_{3,3} \\ u_{3,4} \\ u_{3,5} \\ u_{4,2} \\ u_{4,3} \\ u_{4,4} \\ u_{4,5} \\ u_{5,2} \\ u_{5,3} \\ u_{5,4} \\ u_{5,5}
\end{bmatrix}
=
\begin{bmatrix}
-f_{2,2}h^2k^2 - bh^2 - ck^2 \\
-f_{2,3}h^2k^2 - bh^2 \\
-f_{2,4}h^2k^2 - bh^2 \\
-f_{2,5}h^2k^2 - bh^2 - dk^2 \\
-f_{3,2}h^2k^2 - ck^2 \\
-f_{3,3}h^2k^2 \\
-f_{3,4}h^2k^2 \\
-f_{3,5}h^2k^2 - dk^2 \\
-f_{4,2}h^2k^2 - ck^2 \\
-f_{4,3}h^2k^2 \\
-f_{4,4}h^2k^2 \\
-f_{4,5}h^2k^2 - dk^2 \\
-f_{5,2}h^2k^2 - ah^2 - ck^2 \\
-f_{5,3}h^2k^2 - ah^2 \\
-f_{5,4}h^2k^2 - ah^2 \\
-f_{5,5}h^2k^2 - ah^2 - dk^2
\end{bmatrix}
\quad (10)
$$

where $\alpha = 2(h^2 + k^2)$. Once the system matrix A and the column vector B have been obtained then the SOR Gauss-Seidel [1, 16-17] method with the relaxation parameter $\omega$ can be applied as follows;

$$
x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right), \quad i = 1,2,\ldots,n;\ j = 1,2,\ldots,m. \qquad (11)
$$

Equation (11) computes the numerical solution of (7) at interior nodes denoted by $x_i$. The values are reshaped for consistency to the domain. For the above particular case when f(x,y)=0 the numerical simulation is shown in the following figure for testing purpose.

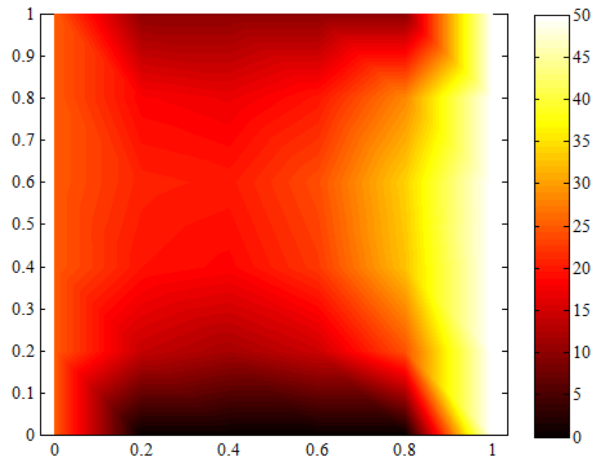


**Figure 3. Numerical simulation for the dimensions and boundary conditions given in the figure (2).**

For the more general case, a concise MATLAB programme was written that automatically transforms the Eq. (6) into the finite difference schemes and then generates the sparse matrix A the column vector B for any dimensions of computational domain with varied boundary conditions, forcing function and error tolerance. The parametric analysis with effect on the convergence behavior in relation to the SOR parameter is discussed in the following Section.

## RESULTS AND DISCUSSION

In order to analyze the convergence behavior of numerical solution of Eq. (11) the different working parameters were changed and the number of iterations were noted at a fixed value of SOR parameter $\omega$. The process was repeated for $\omega = 0.05,\ 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 1.95.$

First of all the effect of mesh spacing parameters h=dx and k=dy was analyzed for $dx = dy = 0.1, 0.2, 0.25, 0.5$ and the results are given in Figure 4. The analysis reveals that for relatively large mesh size the SOR method converges fast as compared to small mesh size. The number of iterations drops down as the values of $\omega$ increases but when $\omega \geq 1.4$ again the number of iterations get increasing. However, for some values of dx=dy the converging iterations remain constant in a subinterval $\omega_1 \subset (0,2)$. This behavior of convergence shows that there is optimum region for the values of $\omega$ which provides the numerical solution in minimum number of iterations. The results were also obtained for unequal mesh spacing in two different ways; first dx was set fixed and dy was varied and then dy was set fixed and dx was varied the results of this setting are exhibited in the Figures (5-6). It is clear form the figures that the convergence of SOR method remain constant after $\omega \geq 0.4$, however for almost small difference in dx and dy the method converges fast.

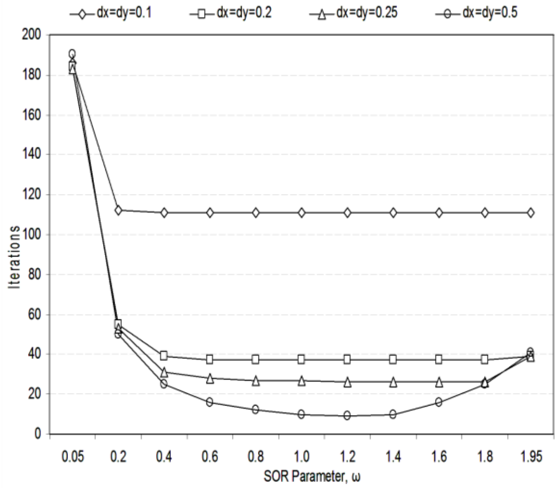


**Figure 4. Variation in the SOR parameter and converging iterations at equal step sizes along both the axes (x and y)**

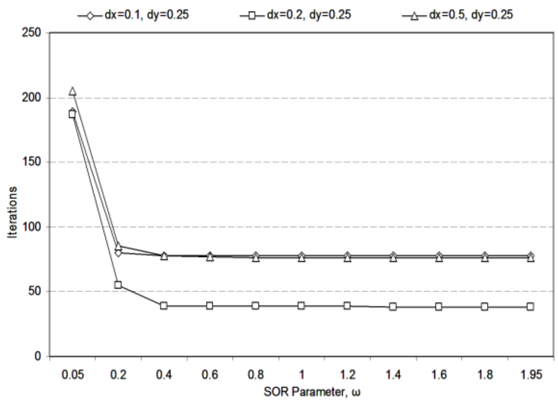


**Figure 5. Variation in the SOR parameter and converging iterations at unequal mesh size along the axes (x and y)**

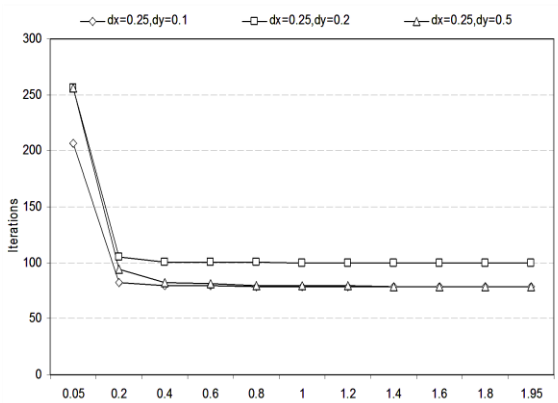


**Figure 6. Variation in the SOR parameter and converging iterations at unequal step sizes along the axis (x and y)**

Similarly, the effect of all four boundary conditions applied on domain and the forcing function f is investigated by varying their values at fixed $\omega$ and the outcomes are shown in Figures (7-11). It was found that there is not much variation in the convergence behavior in terms boundary conditions and forcing function but the analysis exposes that for most of the cases the optimum values of $\omega$ lies in the interval $0.4 \leq \omega \leq 1.8$.

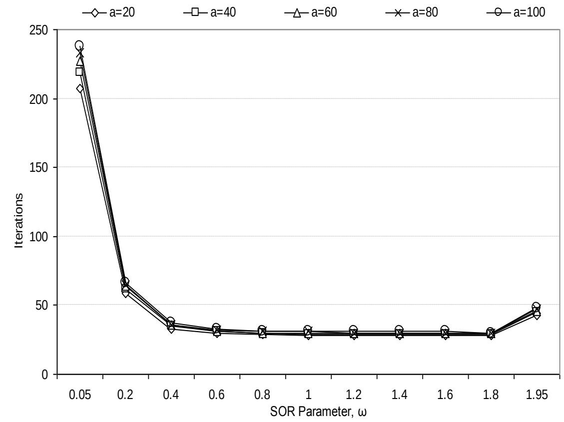


**Figure 7. Variation in the SOR parameter and converging iterations at different values of top boundary condition a.**

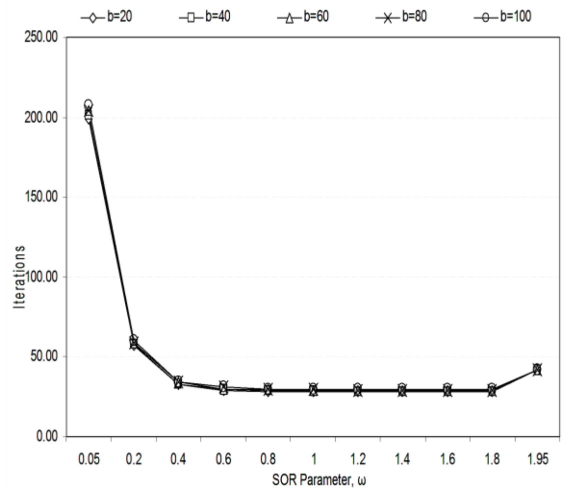


**Figure 8. Variation in the SOR parameter and converging iterations at different values of bottom boundary condition** b.

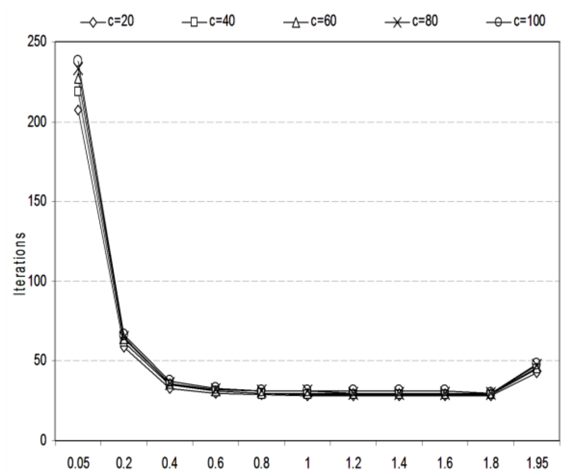


**Figure 9. Variation in the SOR parameter and converging iterations at different values of left boundary condition c.**

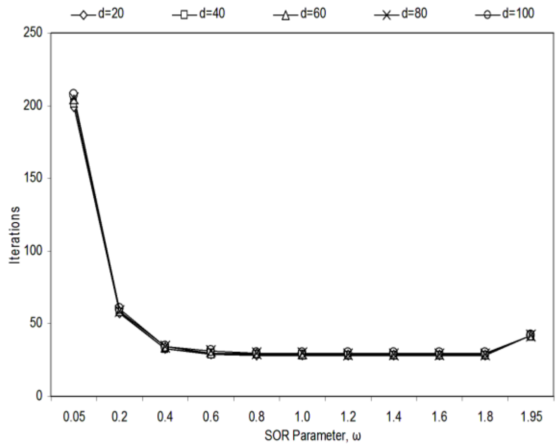


**Figure 10. Variation in the SOR parameter and converging iterations at different values of right boundary condition d.**

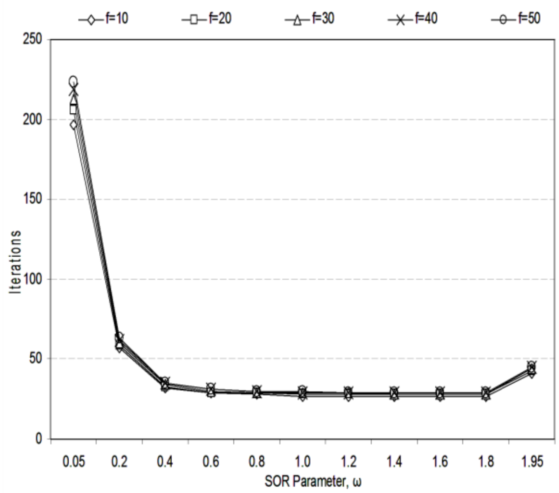


**Figure 11. Variation in the SOR parameter and converging iterations at different values of forcing function f.**

The effect of the domain dimensions and error tolerance have also been analyzed and the results are depicted in Figures (12-14). It can be seen from the figures that the change in domain dimensions do not highly effect the convergence behavior which remain constant after $\omega \geq 0.4$, but the error tolerance appears more sensitive in terms of $\omega$. As the error tolerance increases the number of iterations also increases significantly. Although, there is an optimum interval for the values of $\omega$ that is $0.4 \leq \omega \leq 1.8$. Figur (15) gives an overall analysis of the effect of different parameters on SOR parameter and convergence. The minimum effects from the Figures (4-14) are selected and it is observed that the large difference in the mesh spacing parameters and the error tolerance are more sensitive. Also, in most of the cases the optimum convergence may be achieved when $0.4 \leq \omega \leq 1.8$.

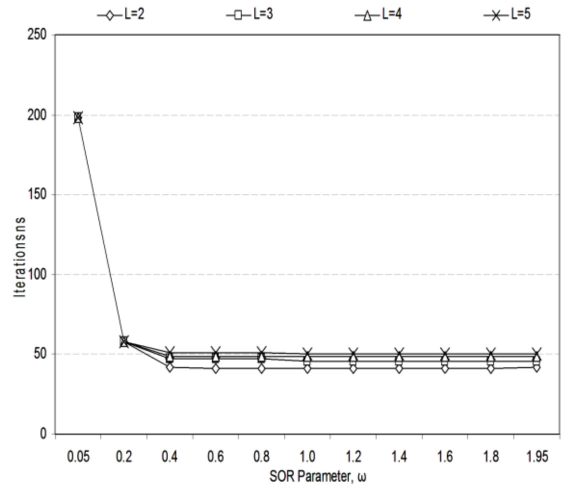


**Figure 12. Variation in the SOR parameter and converging iterations at the different length size L of domain.**

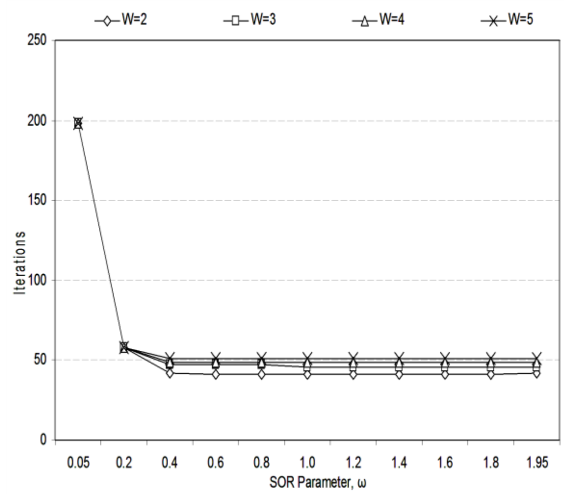


**Figure 13. Variation in the SOR parameter and converging iterations at the different width size W of domain.**

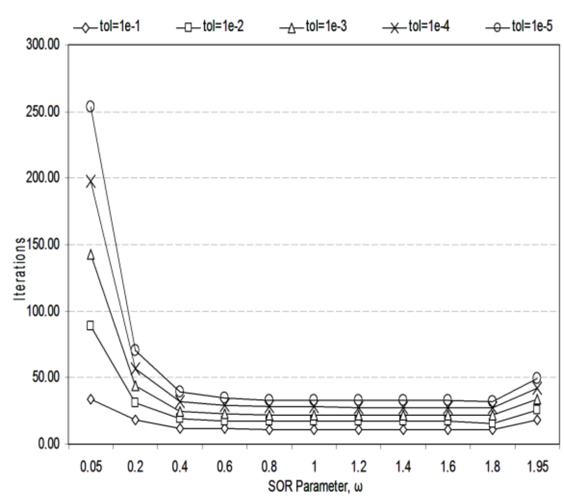


**Figure 14. Variation in the SOR parameter and converging iterations at the different error tolerance**

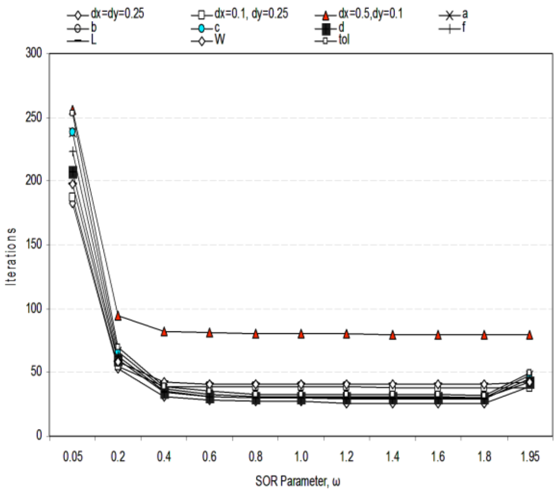


**Figure 15. The overall simultaneous variation in SOR parameter and iterations at different working parameters.**

## CONCLUSION

In this paper a 2D Poisson equation with constant Dirichlet boundary conditions and a constant forcing function was considered. The governing equation was discretized by using finite difference method and solved by SOR Gauss Seidel method to simulate the steady state temperature distribution in a rectangular metallic plate. The objective was to analyze the effect of the different working parameters involved in the numerical solution in relation to SOR parameter. The whole methodology was implemented by writing a MATLAB program and the results were obtained by varying the working parameters in SOR method. The overall analysis has revealed that the mesh spacing parameters and the error tolerance are more sensitive in relation to SOR parameters. For the particular problem solved in this study the optimum values of SOR parameter lies in the interval [0.4, 1.8]. The outcomes of this study provide directions for the optimization of SOR parameter where the empirical relations may be formulated for more general cases with variable boundary conditions, variable forcing function and variable domain dimensions.

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