HAMILTONIAN FORMULATION OF STUECKELBURG FIELD WITH RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

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ABSTRACT: *The Stueckelburg Lagrangian density is reformulated in fractional form in four- dimensional space using Riemann- Liouville fractional derivatives. The fractional Euler-Lagrange equations and the fractional Hamilton's equations are constructed from the fractional Stueckelburg Lagrangian density. The formulation presented and the resulting equations are very similar to those that appear in the field of classical calculus of variations.*

Keywords: Stueckelburg Lagrangian density; Fractional Derivatives; Lagrangian and Hamiltonian Formulation; Fractional Euler-Lagrange Equations.

INTRODUCTION

The Lagrangian formalism is one of the main tools of the description of the dynamics of physical systems including systems with finite (particles) and infinite number of degrees of freedom (fields). It is based on the action principle, which states that the classical motion of a given physical system is such that it extremizes a certain functional of dynamical variables called action. The form of the action determines the equations of motion (Euler-Lagrange equations) of the physical system. Riewe [1, 2] has developed fractional Lagrangian, fractional Hamiltonian, and fractional mechanics. He has shown that Lagrangian with fractional derivative lead directly to equations of motion with non-conservative classical forces. Fractional derivatives have played the significant rule in physics, engineering and applied mathematics [3–10]. Euler–Lagrange equations have been presented for unconstrained and constrained fractional variational problems by Agrawal [11]. The resulting equations are found to be similar to those for variational problems containing integral order derivatives. In other words, the results of fractional calculus of variations reduce to those obtained from traditional fractional calculus of variations when the derivative of fractional order replaced by integral order. This approach is extended to classical fields with fractional derivatives [12]. Baleanu et al [13-16] investigated Euler-Lagrange equations and the fractional Hamilton equations corresponding to a fractional generalization of the equivalent Lagrangians of mechanical and field systems. Rabei et al [17] investigated the classical field with fractional derivatives using the Hamiltonian formalism for discrete and continuous systems.

The aim of this paper is to construct the fractional Hamiltonian equations for the Stueckelburg field within Riemann-Liouville fractional derivatives. The present paper is organized as follows: In Section 2, Riemann – Liouville fractional derivatives are briefly reviewed. In section 3 we propose a new fractional Stueckelburg Lagrangian density. Then in section 4 we obtain fractional Stueckelburg equations using the l Euler-Lagrange equations. Section 5 is devoted to the equations of motion in terms of Hamiltonian density in fractional form. The conclusion is presented in section 6.

1. Mathematical Framework

Fractional calculus is the theory of derivatives and integrals of arbitrary non-integer order. In this section, we formulate the problem in terms of the left and the right Riemann-Liouville (RL) fractional derivatives, which are defined as follows: the left Riemann-Liouvilla fractional derivative, which is denoted by LRLFD, reads as [18]

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(t)d\tau \qquad (1)$$

and the form of right Riemann-Liouville fractional derivative, which is denoted by RRLFD, is given below

$$ {}_tD_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{-d}{dt}\right)^n \int_t^a (t-\tau)^{n-\alpha-1} f(t)d\tau \qquad (2)$$

Here (Γ) represents the gamma function and α is the order of derivative such that ($n-1<\alpha\leq n$) and is not equal zero. If α is an integer, these derivatives become the usual derivatives.

$$ {}_aD_t^\alpha f(t)=\left(\frac{d}{dt}\right)^\alpha f(t) \qquad (3)$$

$$ {}_tD_a^\alpha f(t)=\left(\frac{-d}{dt}\right)^\alpha f(t) \qquad ;\ \alpha=1,2,3\ldots. \qquad (4)$$

By direct calculation we observe that the RL derivative of a constant is not zero, namely

$$ {}_aD_t^\alpha c = c\,\frac{(t-\alpha)^{-\alpha}}{\Gamma(1-\alpha)} \qquad (5)$$

The action of the classical field containing fractional partial derivatives takes the form [19, 20]

$$ S=\int \mathcal{L}(\phi, {}_aD_{x_\mu}^\alpha \phi, {}_{x_\mu}D_b^\beta \phi)d^3x\,dt, \qquad (6)$$

Extremization of this action leads to the fractional Euler-Lagrange equation of the form

$$ \frac{\partial\mathcal{L}}{\partial\phi}+\left[{}_aD_{x_\mu}^\alpha\left(\frac{\partial\mathcal{L}}{\partial\,{}_{x_\mu}D_a^\alpha\phi}\right)+{}_{x_\mu}D_b^\beta\left(\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_\mu}^\beta\phi}\right)\right]=0 \quad (7)$$

For $\alpha=\beta=1$, we have ${}_aD_{x_\mu}^\beta=\partial_\mu$, ${}_{x_\mu}D_a^\alpha=-\partial_\mu$ and the last equation reduces to the standard Euler-Lagrange [21]

$$ \frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi})=0 \qquad (8)$$

The canonical conjugate momentum to ϕ is defined as

$$\pi = \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha \phi} \qquad (9)$$

2. Fractional Stueckelburg Lagrangian Density

The most general form of Lagrangian density for a four-vector field is given by the so-called Stueckelburg Lagrangian density [22] (in SI units where ϵ_0 is the free space permittivity and c is the speed of light),

$$\mathcal{L} = -\frac{\epsilon_0 c^2}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\lambda\epsilon_0 c^2}{2}\left(\partial_\mu A^\mu\right)^2 + \frac{\epsilon_0 c^2\mu^2}{2}\left(A_\mu A^\mu\right) - j_\mu A^\mu \qquad (10)$$

Where $j^\mu = (\rho c, \mathbf{j})$ is the usual four-vector current, λ is a lagrangian multiplier for the Lorentz constraint term, $\mu = 2\pi/\lambda_c = 2\pi mc/h$ is the Compton wave number for photons of mass m, $F^{\mu\nu}$is a four dimension antisymmetric second rank tensor and A^μ is the four-vector potential.

To rewrite the Stueckelburg Lagrangian density in Rie mann – Liouville fractional form use these relations

$$F_{\mu\nu} = \left({}_aD_{x_\mu}^\alpha A_\nu - {}_aD_{x_\nu}^\alpha A_\mu\right) \qquad (11)$$

$$F^{\mu\nu} = \left({}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x^\nu}^\alpha A^\mu\right) \qquad (12)$$

$$F_{\mu\nu}F^{\mu\nu} = 2\left[{}_aD_{x_\mu}^\alpha A_\nu\, {}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x_\nu}^\alpha A_\mu\, {}_aD_{x^\nu}^\alpha A^\mu\right] \qquad (13)$$

Where $\mu = 0\,, i\,; i = 1{,}2{,}3$ and $\nu = 0\,,\ j\,; j = 1{,}2{,}3$.

Expand μ, ν in terms of 0, i and 0, j, respectively, and use these relativistic notations

$A^\nu = (\phi, \boldsymbol{A})$ $\qquad A_\nu = (\phi, -\boldsymbol{A})$

${}_aD_{x_\mu}^\alpha = (\,{}_aD_t^\alpha, {}_aD_{x_i}^\alpha)$ $\qquad {}_aD_{x^\mu}^\alpha = (\,{}_aD_t^\alpha, -{}_aD_{x^i}^\alpha)$

Thus, the fractional Stueckelburg Lagrangian density in Riemann – Liouville fractional derivative becomes

$$\mathcal{L} = -\frac{\epsilon_0 c^2}{2}\begin{bmatrix} -{}_aD_t^\alpha A^j\, {}_aD_t^\alpha A^j + {}_aD_t^\alpha A^j\, {}_aD_{x^j}^\alpha \emptyset \\ -{}_aD_{x^i}^\alpha \emptyset\, {}_aD_{x^i}^\alpha \emptyset + {}_aD_{x^i}^\alpha \emptyset\, {}_aD_t^\alpha A^i + \\ {}_aD_{x^i}^\alpha A^j\, {}_aD_{x^i}^\alpha A^j - {}_aD_{x^i}^\alpha A^j\, {}_aD_{x^j}^\alpha A^i \end{bmatrix} - \frac{\lambda\epsilon_0 c^2}{2}\left({}_aD_t^\alpha \emptyset + {}_aD_{x_i}^\alpha A^i\right)^2 + \frac{\epsilon_0 c^2\mu^2}{2}\left(\emptyset^2 - {A^i}^2\right) - j_0\emptyset - j_i A^i \qquad (14)$$

or

$$\mathcal{L} = -\frac{\epsilon_0 c^2}{2}\begin{bmatrix} -{}_aD_t^\alpha A^j\, {}_aD_t^\alpha A^j + {}_aD_t^\alpha A^j\, {}_aD_{x^j}^\alpha \emptyset \\ -{}_aD_{x^i}^\alpha \emptyset\, {}_aD_{x^i}^\alpha \emptyset + {}_aD_{x^i}^\alpha \emptyset\, {}_aD_t^\alpha A^i + \\ {}_aD_{x^i}^\alpha A^j\, {}_aD_{x^i}^\alpha A^j - {}_aD_{x^i}^\alpha A^j\, {}_aD_{x^j}^\alpha A^i \end{bmatrix} - \frac{\lambda\epsilon_0 c^2}{2}\left({}_aD_t^\alpha \emptyset\, {}_aD_t^\alpha \emptyset - 2\, {}_aD_t^\alpha \emptyset\, {}_aD_{x^i}^\alpha A^i + {}_aD_{x^i}^\alpha A^i\, {}_aD_{x^i}^\alpha A^i\right) + \frac{\epsilon_0 c^2\mu^2}{2}\left(\emptyset^2 - {A^i}^2\right) - j_0\emptyset - j_i A^i \qquad (15)$$

3. *Fractional form of Euler-Lagrange Equations of* Stueckelburg *Lagrangian Density*

Let us start with the definition of fractional Stueckelburg Lagrangian density and use the generalization formula of Euler – Lagrange equation (7) to obtain the equations of motion from Stueckelburg Lagrangian density.

Take the first field variable $\emptyset$, then

$$\frac{\partial \mathcal{L}}{\partial \phi} - {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial({}_aD_t^\alpha \phi)} - {}_aD_{x^i}^\alpha \frac{\partial \mathcal{L}}{\partial({}_aD_{x^i}^\alpha \phi)} = 0 \qquad (16)$$

Calculating these derivatives yields to

$$\frac{\partial \mathcal{L}}{\partial \phi} = \epsilon_0 c^2\mu^2\phi - j_0 \qquad (17)$$

$$\frac{\partial \mathcal{L}}{\partial({}_aD_t^\alpha \phi)} = -\lambda\epsilon_0 c^2\left({}_aD_t^\alpha \phi - {}_aD_{x^i}^\alpha A^i\right) \qquad (18)$$

$$\frac{\partial \mathcal{L}}{\partial({}_aD_{x^i}^\alpha \phi)} = -\epsilon_0 c^2\left({}_aD_t^\alpha A^i - {}_aD_{x^i}^\alpha \phi\right) \qquad (19)$$

Substituting equations (17, 18, and 19) in equation (16) we get

$$j_0 - \epsilon_0 c^2\mu^2\phi = \lambda\epsilon_0 c^2\, {}_aD_t^\alpha\left({}_aD_t^\alpha \phi - {}_aD_{x^i}^\alpha A^i\right) + \epsilon_0 c^2\, {}_aD_{x^i}^\alpha\left({}_aD_t^\alpha A^i - {}_aD_{x^i}^\alpha \phi\right) \qquad (20)$$

This represents the first non- homogeneous equation in fractional form.

Now use the general formula (7) to obtain other equations of motion from the other fields' variables A^i and A^j.

$$\frac{\partial \mathcal{L}}{\partial A^i} - {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial({}_aD_t^\alpha A^i)} - {}_aD_{x^i}^\alpha \frac{\partial \mathcal{L}}{\partial({}_aD_{x^i}^\alpha A^i)} = 0 \qquad (21)$$

Calculating these derivatives yields to

$$\frac{\partial \mathcal{L}}{\partial A^i} = -\epsilon_0 c^2\mu^2 A^i - j_i \qquad (22)$$

$$\frac{\partial \mathcal{L}}{\partial({}_aD_t^\alpha A^i)} = \frac{-\epsilon_0 c^2}{2}\left({}_aD_{x^i}^\alpha \phi\right) \qquad (23)$$

$$\frac{\partial \mathcal{L}}{\partial({}_aD_{x^i}^\alpha A^i)} = -\lambda\epsilon_0 c^2\left(-{}_aD_t^\alpha \phi + {}_aD_{x^i}^\alpha A^i\right) \qquad (24)$$

Substituting equations (22, 23, and 24) in equation (21) we get

$$-\epsilon_0 c^2\mu^2 A^i - j_i + \frac{\epsilon_0 c^2}{2}\, {}_aD_t^\alpha\left({}_aD_{x^i}^\alpha \phi\right) + \lambda\epsilon_0 c^2\, {}_aD_{x^i}^\alpha\left(-{}_aD_t^\alpha \phi + {}_aD_{x^i}^\alpha A^i\right) = 0 \qquad (25)$$

And

$$\frac{\partial\mathcal{L}}{\partial A^j} - {}_aD_t^\alpha \frac{\partial\mathcal{L}}{\partial({}_aD_t^\alpha A^j)} - {}_aD_{x^i}^\alpha \frac{\partial\mathcal{L}}{\partial({}_aD_{x^i}^\alpha A^j)} = 0 \quad (26)$$

$$\frac{\partial\mathcal{L}}{\partial A^j} = 0 \quad (27)$$

$$\frac{\partial\mathcal{L}}{\partial({}_aD_t^\alpha A^j)} = \frac{-\epsilon_0 c^2}{2}\left(-2{}_aD_t^\alpha A^j + {}_aD_{x^j}^\alpha \phi\right) \quad (28)$$

$$\frac{\partial\mathcal{L}}{\partial({}_aD_{x^i}^\alpha A^j)} = \frac{-\epsilon_0 c^2}{2}\left(-{}_aD_{x^j}^\alpha A^i + 2{}_aD_{x^i}^\alpha A^j\right) \quad (29)$$

Substituting equations (26, 27, and 28) in equation (25) we get

$$\frac{\epsilon_0 c^2}{2}{}_aD_t^\alpha\left(-2{}_aD_t^\alpha A^j + {}_aD_{x^j}^\alpha \phi\right) + \frac{\epsilon_0 c^2}{2}{}_aD_{x^i}^\alpha\left(-{}_aD_{x^j}^\alpha A^i + 2{}_aD_{x^i}^\alpha A^j\right) = 0 \quad (30)$$

Adding equations (25) and (30) to get

$$j_i + \epsilon_0 c^2 \mu^2 A^i = \epsilon_0 c^2{}_aD_t^\alpha\left({}_aD_{x^i}^\alpha \phi - {}_aD_t^\alpha A^j\right) + \lambda\epsilon_0 c^2{}_aD_{x^i}^\alpha\left(-{}_aD_t^\alpha \phi + {}_aD_{x^i}^\alpha A^i\right) + \frac{\epsilon_0 c^2}{2}{}_aD_{x^i}^\alpha\left(-{}_aD_{x^j}^\alpha A^i + 2{}_aD_{x^i}^\alpha A^j\right) \quad (31)$$

This represents the second non- homogeneous equation in fractional form.

If α goes to 1, Eqs. (20) and (31) go to the standard equations.

4. *Equations of Motion in terms of Hamiltonian Formulation*

To construct the fractional Hamiltonian equations within Riemann – Liouville fractional derivative from fractional Stueckelburg Lagrangian density, we consider the Lagrangian density to be a function of field variables and its fractional derivatives with respect to space and time as:

$$\mathcal{L}=\mathcal{L}\left(\emptyset, A^i, A^j, {}_aD_t^\alpha A^j, {}_aD_t^\alpha A^i, {}_aD_t^\alpha \emptyset, {}_aD_{x^i}^\alpha A^j, {}_aD_{x^j}^\alpha A^i, {}_aD_{x^i}^\alpha \emptyset, t\right) \quad (31)$$

We introduce the conjugate momenta as

$$\begin{cases} \pi_{\alpha_\emptyset} = \dfrac{\partial\mathcal{L}}{\partial({}_aD_t^\alpha \emptyset)} \\ \pi_{\alpha_{A^i}} = \dfrac{\partial\mathcal{L}}{\partial({}_aD_t^\alpha A^i)} \\ \pi_{\alpha_{A^j}} = \dfrac{\partial\mathcal{L}}{\partial({}_aD_t^\alpha A^j)} \end{cases} \quad (32)$$

and the Hamiltonian depending on the fractional time derivatives reads as

$$\mathcal{H} = \pi_{\alpha_\emptyset}{}_aD_t^\alpha \phi + \pi_{\alpha_{A^i}}{}_aD_t^\alpha A^i + \pi_{\alpha_{A^j}}{}_aD_t^\alpha A^j - \mathcal{L} \quad (33)$$

Take the total differential of both sides we get

$$d\mathcal{H} = \begin{cases} \pi_{\alpha_\emptyset} d({}_aD_t^\alpha \phi) + d(\pi_{\alpha_\emptyset}){}_aD_t^\alpha \phi + \pi_{\alpha_{A^i}} d({}_aD_t^\alpha A^i) \\ +(d\pi_{\alpha_{A^i}}){}_aD_t^\alpha A^i + \pi_{\alpha_{A^j}} d({}_aD_t^\alpha A^j) + d(\pi_{\alpha_{A^j}}){}_aD_t^\alpha A^j \\ -\dfrac{\partial\mathcal{L}}{\partial({}_aD_t^\alpha A^i)} d({}_aD_t^\alpha A^i) - \dfrac{\partial\mathcal{L}}{\partial({}_aD_t^\alpha A^j)} d({}_aD_t^\alpha A^j) \\ -\dfrac{\partial L}{\partial({}_aD_t^\alpha \emptyset)} d({}_aD_t^\alpha \emptyset) - \dfrac{\partial L}{\partial t} dt - \dfrac{\partial L}{\partial \phi} d\phi \\ -\dfrac{\partial L}{\partial A^j} dA^j - \dfrac{\partial\mathcal{L}}{\partial A^i} dA^i - \dfrac{\partial\mathcal{L}}{\partial({}_aD_{x^j}^\alpha A^i)} d({}_aD_{x^j}^\alpha A^i) \\ -\dfrac{\partial\mathcal{L}}{\partial({}_aD_{x^i}^\alpha A^j)} d({}_aD_{x^i}^\alpha A^j) - \dfrac{\partial\mathcal{L}}{\partial({}_aD_{x^i}^\alpha \emptyset)} d({}_aD_{x^i}^\alpha \emptyset) \end{cases} \quad (34)$$

Substituting the values of conjugate momenta from equation (32), we get

$$d\mathcal{H} = \begin{cases} d(\pi_{\alpha_\emptyset}){}_aD_t^\alpha \phi + (d\pi_{\alpha_{A^i}}){}_aD_t^\alpha A^i \\ +d(\pi_{\alpha_{A^j}}){}_aD_t^\alpha A^j - \frac{\partial\mathcal{L}}{\partial t} dt - \frac{\partial\mathcal{L}}{\partial \phi} d\phi \\ -\frac{\partial\mathcal{L}}{\partial A^j} dA^j - \frac{\partial\mathcal{L}}{\partial A^i} dA^i - \frac{\partial\mathcal{L}}{\partial({}_aD_{x^j}^\alpha A^i)} d({}_aD_{x^j}^\alpha A^i) \\ -\frac{\partial\mathcal{L}}{\partial({}_aD_{x^i}^\alpha A^j)} d({}_aD_{x^i}^\alpha A^j) - \frac{\partial\mathcal{L}}{\partial({}_aD_{x^i}^\alpha \emptyset)} d({}_aD_{x^i}^\alpha \emptyset) \end{cases} \quad (35)$$

But the Hamiltonian is function of the form

$$\mathcal{H} = \mathcal{H}\left(\emptyset, A^i, A^j, t, \pi_{\alpha_{A^i}}, \pi_{\alpha_{A^j}}, {}_aD_{x^i}^\alpha \emptyset, {}_aD_{x^i}^\alpha A^j, {}_aD_{x^j}^\alpha A^i\right) \quad (36)$$

So the total differential of Hamiltonian function reads as:

$d\mathcal{H} =$

$$\begin{cases} \frac{\partial\mathcal{H}}{\partial\pi_{\alpha_\emptyset}}d\pi_{\alpha_\emptyset}+\frac{\partial\mathcal{H}}{\partial\left({}_aD^\alpha_{x_j}\overline{\Psi}\right)}d\left({}_aD^\alpha_{x_j}\overline{\Psi}\right) \\ +\frac{\partial\mathcal{H}}{\partial\pi_{\alpha_{A^j}}}d\pi_{\alpha_{A^j}}+\frac{\partial\mathcal{H}}{\partial\pi_{\alpha_{A^i}}}d\pi_{\alpha_{A^i}}+\frac{\partial\mathcal{H}}{\partial A^j}dA^j \\ +\frac{\partial\mathcal{H}}{\partial A^i}dA^i+\frac{\partial\mathcal{H}}{\partial\emptyset}d\emptyset+\frac{\partial\mathcal{H}}{\partial t}dt+ \\ \frac{\partial\mathcal{H}}{\partial\left({}_aD^\alpha_{x^i}\emptyset\right)}d\left({}_aD^\alpha_{x^i}\emptyset\right)+\frac{\partial\mathcal{H}}{\partial\left({}_aD^\alpha_{x^i}A^j\right)}d\left({}_aD^\alpha_{x^i}A^j\right) \\ +\frac{\partial\mathcal{H}}{\partial\left({}_aD^\alpha_{x^j}A^i\right)}d\left({}_aD^\alpha_{x^j}A^i\right) \end{cases} \quad (37)$$

Comparing equations (36) and (37), we get the following Hamilton's equations of motion:

$$\begin{cases} \frac{\partial\mathcal{H}}{\partial t}=-\frac{\partial\mathcal{L}}{\partial t}, & \frac{\partial\mathcal{H}}{\partial\pi_{\alpha_{A^j}}}={}_aD^\alpha_t A^j \\ \frac{\partial\mathcal{H}}{\partial\pi_{\alpha_{A^i}}}={}_aD^\alpha_t A^i, & \frac{\partial\mathcal{H}}{\partial\pi_{\alpha_{A^i}}}={}_aD^\alpha_t A^i \end{cases} \quad (38)$$

$$\begin{cases} \frac{\partial\mathcal{H}}{\partial\left({}_aD^\alpha_{x^i}\emptyset\right)}=-\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_{x^i}\emptyset\right)} \;. \\ \frac{\partial\mathcal{H}}{\partial\left({}_aD^\alpha_{x^i}A^j\right)}=-\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_{x^i}A^j\right)} \\ \frac{\partial\mathcal{H}}{\partial\left({}_aD^\alpha_{x^j}A^i\right)}=-\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_{x^j}A^i\right)} \end{cases} \quad (39)$$

$$\begin{cases} \frac{\partial\mathcal{H}}{\partial\emptyset}=-\frac{\partial\mathcal{L}}{\partial\phi} \\ \frac{\partial\mathcal{H}}{\partial A^i}=-\frac{\partial\mathcal{L}}{\partial A^i} \\ \frac{\partial\mathcal{H}}{\partial A^j}=-\frac{\partial\mathcal{L}}{\partial A^j} \end{cases} \quad (40)$$

The final group of these equations can be rewritten using the Euler-Lagrange equation (7) to get equations of motion in fractional form.

$$\frac{\partial\mathcal{H}}{\partial\emptyset}=-{}_aD^\alpha_t\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_t\phi\right)}-{}_aD^\alpha_{x^i}\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_{x^i}\phi\right)} \quad (41)$$

Finding these derivatives, then we get

$$\frac{\partial\mathcal{H}}{\partial\emptyset}=-{}_aD^\alpha_t[-\lambda\epsilon_0c^2({}_aD^\alpha_t\phi-{}_aD^\alpha_{x^i}A^i)] - {}_aD^\alpha_{x^i}[-\epsilon_0c^2({}_aD^\alpha_tA^i-{}_aD^\alpha_{x^i}\phi)]$$

Rearrange this equation

$$j_0-\epsilon_0c^2\mu^2\phi = \lambda\epsilon_0c^2\,{}_aD^\alpha_t\left({}_aD^\alpha_t\phi-{}_aD^\alpha_{x^i}A^i\right)+\epsilon_0c^2\,{}_aD^\alpha_{x^i}\left({}_aD^\alpha_tA^i-{}_aD^\alpha_{x^i}\phi\right) \quad (42)$$

This is the first non- homogeneous equation in fractional form.

Now take other fields variables A^i, A^j

$$\frac{\partial\mathcal{H}}{\partial A^i}=-{}_aD^\alpha_t\frac{\partial\mathcal{L}}{\partial({}_aD^\alpha_tA^j)}-{}_aD^\alpha_{x^i}\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_{x^i}A^j\right)} \quad (43)$$

$$\epsilon_0c^2\mu^2A^i+j_i=\frac{\epsilon_0c^2}{2}\,{}_aD^\alpha_t\left({}_aD^\alpha_{x^i}\phi\right) + \lambda\epsilon_0c^2\,{}_aD^\alpha_{x^i}\left(-{}_aD^\alpha_t\phi+{}_aD^\alpha_{x^i}A^i\right) \quad (44)$$

And

$$\frac{\partial\mathcal{H}}{\partial A^j}=-{}_aD^\alpha_t\frac{\partial\mathcal{L}}{\partial({}_aD^\alpha_tA^j)}-{}_aD^\alpha_{x^i}\frac{\partial\mathcal{L}}{\partial\left({}_aD^\alpha_{x^i}A^j\right)} \quad (45)$$

$$0=\frac{\epsilon_0c^2}{2}\,{}_aD^\alpha_t\left(-2\,{}_aD^\alpha_tA^j+{}_aD^\alpha_{x^j}\phi\right) +\frac{\epsilon_0c^2}{2}\,{}_aD^\alpha_{x^i}\left(-{}_aD^\alpha_{x^j}A^i+2\,{}_aD^\alpha_{x^i}A^j\right) \quad (46)$$

Add equations (44) and (46) to obtain

$$j_i+\epsilon_0c^2\mu^2A^i = \epsilon_0c^2\,{}_aD^\alpha_t\left({}_aD^\alpha_{x^i}\phi-{}_aD^\alpha_tA^j\right)+\lambda\epsilon_0c^2\,{}_aD^\alpha_{x^i}\left(-{}_aD^\alpha_t\phi+{}_aD^\alpha_{x^i}A^i\right) + \frac{\epsilon_0c^2}{2}\,{}_aD^\alpha_{x^i}\left(-{}_aD^\alpha_{x^j}A^i+2\,{}_aD^\alpha_{x^i}A^j\right) \quad (47)$$

This represents the second non- homogeneous equation in fractional form.

5. CONCLUSION

The Stueckelburg Lagrangian density is reformulated using fractional calculus with left-right Riemann-Liouville fractional derivatives. The derivation of the usual Euler-Lagrange equations of motion for classical field has extended to the case the Lagrangian contains fractional derivatives of the field. This method has been applied with the variational principle to obtain the corresponding fractional Euler-Lagrange equations from the fractional Stueckelburg Lagrangian density. The fractional Hamilton's equations are obtained for Stueckelburg Lagrangian density. Our results are the same as those derived by using the formulation of Euler- Lagrange equations. The classical results are reobtained when α goes to 1.

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