REVISITING GAUSSIAN PROFILE FOR FOURIER TRANSFORM AND USING THE FRACTIONAL FOURIER TRANSFORM FOR WAVELETS

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ABSTRACT: *The Mexican hat and Morlet wavelets in a Young's double experiment require quantum mechanical calculations, the use of quantum wave optics instead of classical wave optics. With Gaussian profile of Fourier transform and its change with curvature of time, stretching, twisting and twigging of quanta occurs. The stretching is evidenced with left hand skewness of Gaussian function. The stretched quanta is twisted and then twigged. Mathematical relations for stretching, twisting and twigging are obtained. The Mexican hat and Morlet wavelets are the manifestations of twigging angle,* $\alpha\left(0 \le \alpha \le \frac{\pi}{2}\right)$ *and followed by skewness Gaussian profile with adiabatic and harmonic perturbation, respectively. With twisting, quantization of events,.i.e.,* $\hbar t$ *occurs both in over damped and under damped quantum states, 0.1*$\le nf \le 0.9$ *with algorithmic profiles, respectively. Resonant state (bright and dark fringes) are produced by momentum impact,* $\hbar k$ *as a manifestation of diffraction and then superposition (quantum wave optics is to be used for optical path difference in terms of* $r_{op} \equiv \delta(R - R_0)$*). Such a behavior of quantum wave optics leads to "quantum entanglement" in the Wiener space and with inversion symmetry which can be extended to indefinite distances in the reciprocal space (non-locality),.i.e., the Wigner space. The Fractional Fourier transform is used as a limit to Gaussian profile of the Fourier transform in* (t, ω) *plane with twisting angle* $0 \le \alpha \le \frac{\pi}{2}$ *and the quantization of events with diverse frequencies of the Mexican hat and Morlet wavelets but integrated in a braid/wire of quanta.Wavelets also need superposition followed by quantum wave optics and indeed 'quantum entanglement ' with fractional states,0.1*$0.1 \le n_f \le 0.9$.

Key words: Morlet wavelet; Fractional Fourier Transform; Mexican wavelet; Gaussian profile

INTRODUCTION

The origin of Mexican hat and Morlets wavelets produced due to superposed transverse waves is dealtwith thought provoking experiments. Consider a ripple tank filled with water. Drop two identical bodies (having same mass and locations in the form of continuous wavefronts which interfere (superpose) both constructively and destructively. In between these patterns (constructive and destructive), wavefronts are broken. These broken wavefronts will obey Doppler Effect and have the shape of Mexican hat and Morlet wavelets [1]. These transverse waves or wavelets also obey directions cosines, .i.e., spreading of energy (wave) or of wavelets on a plane surface. Thus the dot products of two transverse waves and of wavelets follow the direction cosine. This shows that Mexican hat and Morlets do not obey curvature of space, .i.e., $\frac{\partial}{\partial r}$. Pont, these wavelets obey curvature of time, .i.e.,$\frac{\partial}{\partial t}$. Fourier transform fails for these wavelets at $\alpha = \omega t \le \frac{\pi}{2}$. The wavelets whichare accompanied with Doppler Effect, having minimum energy are broken from wavefronts because twisting is restricted because the cross product of the particles of two suchwaves A and B is zero. This is why equation (1) and (2) in [1] by Prasad et al.

$$\bar{\phi}(\omega) = (\mathfrak{F}\phi)(\omega) = \frac{1}{\sqrt{2\pi}}\int e^{-it\omega}\,\phi(t)dt, \omega\epsilon\mathbb{R} \qquad (1)$$

$$\phi(t) = (\mathfrak{F}^{-1}\phi)(t) = \frac{1}{\sqrt{2\pi}}\int e^{it\omega}\bar{\phi}(\omega)d\omega, t\epsilon\mathbb{R} \qquad (2)$$

follow the Fourier transform based on curvature of time or on$\alpha = \omega t$. The spread of wave energy (transverse) on a plane for superposition of wavelets follow a free momentum space (linear). But we are treating FRFT (Fractional Fourier Transform) for these wavelets. In such a trivial solution, momentum space for these particles (wavelets) are quantized. In FRFT, we consider Gaussian wave packet for Mexican hat and Morlet wavelets, respectively. Our paradox for such a problem could be envisaged with a Young's double slit experiment for photons. Photons show metamorphic behaviour. The experiments of Compton Effect and photoelectric effect show the particle behaviour of light(photons) whereas the experiments of physical optics (Interference, diffraction and polarization) show the wave nature of light. The same is true for electromagnetic waves (Transverse waves). We have to modify the Parseval's formula for fractional Fourier transform (FRFT).We consider basicformulasfromrelativelynewbooksonquantumphysics [2, 3,4].

Almeida [5] defines the fractional Fourier transform (FRFT) of a function$x(t)$, with an angle α($t$is the time and $\omega$is the frequency) as

$$\mathfrak{F}_\alpha[\phi(t)] = \bar{\phi}_\alpha(\omega) = \int_{-\infty}^{\infty}\phi(t)K_\alpha(t,\omega)dt$$

$$= \begin{cases} \sqrt{\dfrac{1 - icot\alpha}{2\pi}}\, e^{i\frac{\omega^2}{2}\cot\alpha}\displaystyle\int_{-\infty}^{\infty}\phi(t)e^{i\frac{t^2}{2}\cot\alpha - i\omega t\csc\alpha}dt \\ \text{if } \alpha \text{ is not a multiple of } \pi \\ \phi(t)\text{If } \alpha \text{ is a multiple of } 2\pi \\ \phi(-t) \text{ if } \alpha + \pi \text{ is a multiple of } 2\pi \end{cases} \qquad (3)$$

2 Theory

Wavelets are a family of functions constructed from translation and dilation of a functionψ, which is called the mother wavelet, and defined in [1],

$$\psi_{b,a}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), \; b \in \mathbb{R}, a > 0 \qquad (4)$$

Where $a$is called the scaling parameter, which measures the degree of compression or scale, and b is translate parameter, which determines the time location of wavelet. Shi et al in their article [6], defined the fractional mother wavelet as

$$\psi_{b,a,\alpha}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)e^{-\frac{i}{2}(t^2-b^2)\cot\alpha} \quad (5)$$

The Fourier transform $\psi(r)$is revisited for a plane wave, the amplitude function defined as

$$a(k) = \begin{cases} \frac{1}{\varepsilon} - \frac{\varepsilon}{2} \le k \le \frac{\varepsilon}{2} \\ 0 \qquad |k| > \frac{\varepsilon}{2} \end{cases} \quad (6)$$

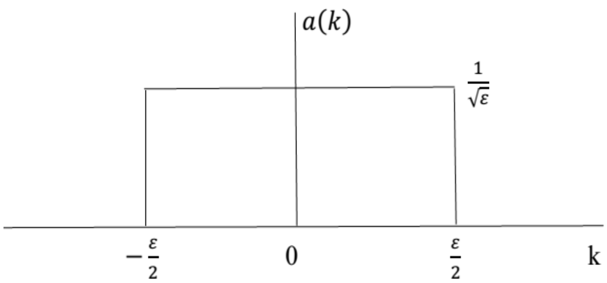

Figure 1: Amplitude function for a plane wave

This amplitude function, the Square of which is the intensity, may be of photons, has the Fourier Transform

$$\psi(r) = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}\frac{1}{\sqrt{\varepsilon}}e^{ikr}dk = \sqrt{\frac{2}{\pi\varepsilon}}\frac{sin\left(\varepsilon\frac{r}{2}\right)}{r} \quad (7)$$

Parseval's formula for equation (7) is

$$\int_{-\infty}^{\infty}|a(k)|^2dk = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}}\frac{1}{\varepsilon}dk = 1 = \int_{-\infty}^{\infty}|\psi(r)|^2dr \quad (8)$$

In our example we are considering diffracted wavefronts (plane waves) from two identical slits, which superpose both constructively and destructively (Young's double slit experiment). Wavelets are apparently broken pieces of wavefronts which are on either sides and in the vicinity of bright fringes of photons (Fresnel Diffraction with Young's double slit experiment) as well as for dark fringes. For diffraction pattern n, .i.e., we get for bright and dark fringes, the optical path difference $n\lambda$; $n = 0, 1, 2, 3$ ...(Constructive interference)$\frac{n}{2}\lambda$; $n = 1,3,5$ ... (Destructive interference)

Equation (7) is to be modified for superposition of two waves (transverse wave for photons). Parseval's formula, .i.e.,

$$\int \psi|(r)|^2\, dr = \int a|(p)|^2\, dp$$

is to be modified. We consider linear combination of two functions $\psi(r)$ and $\phi(r)$ whose momentum representation are $a(p)$ and$b(p)$. Remember that we are dealing with superposition of two light waves which can be dealt only with quantum physical optics or quantum optics. The linear combination of $\psi(r)$ and $\phi(r)$ will follow a complex constant. Moreover$\psi(r)$ and $\phi(r)$ are no more defined in trigonometric space but indeed in hyperbolic space.

$$\int|\psi(r)+\lambda\phi(r)|^2dr = \int[|\psi(r)+ \lambda\psi^*(r)\phi(r)+\lambda^*\phi^*(r)\psi(r) + |\lambda|^2|\phi(r)|^2|^2]dr \quad (9)$$

$$\int|a(p)+\lambda b(p)|^2dp = \int[|a(p)+ \lambda a^*(p)b(p)+\lambda^* b^*(p)a(p) + |\lambda|^2|b(p)|^2|^2]\,dp \quad (10)$$

And therefore,

$$\lambda\int\psi^*(r)\phi(r)dr + \lambda^*\int\phi^*(r)\psi(r)dr = \lambda\int a^*(p)b(p)dp + \lambda^*\int b^*(p)a(p)dp \quad (11)$$

Which can only be true for every complex value of constantλ

$$\int\psi^*(r)\phi(r)dr = \int a^*(p)b(p)dp$$

(only in the resonant states) (12)

where$\psi(r), \psi^*(r), \phi(r)$are eigen functions and indeed operators $\psi^*(r)$, and $\phi^*(r)$, follow inversion symmetry with $\frac{a}{\lambda}$, where λ is complex constant and an operator. The Fourier transform can be written as

$$\psi(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a(k)e^{ikr}dk;\ \phi(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}b(k)e^{ikr}dk \quad (13)$$

$$a(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(r)e^{-ikr}dr;\ b(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(r)e^{-ikr}dr \quad (14)$$

Where $p = \hbar k$ is the quantum mechanical momentum. It is obtained from de Broglie wave hypothesis, .i.e.; $p = \frac{\hbar}{\lambda}$ where □ is quantum action and Plank's constant.

$$\psi^*(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a^*(k)e^{-ikr}\,dk\ ;\phi^*(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}b^*(k)e^{-ikr}\,dk \ 15)$$

$$a^*(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi^*(r)e^{-ikr}\,dr\ ; b^*(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi^*(r)e^{-ikr}\,dr \ 16)$$

$|\psi(r)|^2, |\psi^*(r)|^2, |\phi(r)|^2, |\phi^*(r)|^2$, etc. can be verified with the following relations:

$$\int_{-\infty}^{\infty} |\psi(r)|^2\, dr = \frac{2}{\pi\varepsilon} \int_{-\infty}^{\infty} \frac{sin^2\left(\varepsilon\frac{r}{2}\right)}{r^2} dr = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{sin^2 r}{r^2} dr$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{sinr}{r} dr$$
$$= 1 \qquad (17)$$

The inverse transform of equation (12), say for example, for $\psi(r)$ is

$$a(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi\varepsilon}} \frac{sin\left(\varepsilon\frac{r}{2}\right)}{r} e^{-ikr} dr \qquad (18)$$

Equation (18) is obtained with the help of residues and poles (complex integration). We observed experimentally that the amplitude function, for example $a(k)$ has the width ε while eigenfunction $\psi(r)$ has a width of order $\frac{1}{\varepsilon}$. Using the Heisenberg uncertainty principle, .i.e.

$$< \Delta p >< \Delta r > \approx h \qquad (19)$$
$$< \Delta \frac{h}{\lambda} >< \Delta r > \approx \hbar \Rightarrow < \Delta \frac{1}{\lambda} >< \Delta r > \approx 1$$
$$\Rightarrow < \Delta p >< \Delta r > \approx 1$$

The uncertainty principle in equation (19) follows the semi classical wave mechanics, .i.e., $k = \frac{2\pi}{\lambda}$ or $\frac{1}{\lambda}$, the propagation vectors,k in quantum physics are

$$k_{free} = \frac{\sqrt{2mE}}{\hbar}, \quad k_{quantized} = \frac{\sqrt{2m(E-v)}}{\hbar} \qquad (20)$$

The position, momentum and energy operators in quantum physics are defined as

$$r_{op} = \delta(r - r_0), p_{op} = -i\nabla\hbar = -i\hbar\frac{\partial}{\partial r}, E_{op} = \frac{-\hbar}{i}\frac{\partial}{\partial t} = i\hbar\frac{\partial}{\partial t} \qquad (21)$$

The scaling and translation parameters of wavelets inhibit the use of the fractional Fourier transform (FRFT) because the the degree of superposition and time location of wavelets (Maxican hat and Morletwavelets) fail in the (t, ω) complex plane. For this we have to apply conformal mapping.

Considering the Gaussian function we have to use fractional Fouriertransform (FRFT)

$$\psi(r) = \frac{1}{\sqrt{\sigma\sqrt{\pi}}} e^{-\left(\frac{r^2}{2\sigma^2}\right)} (22)$$

The Fourier transform of Gaussian function is

$$a(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma\sqrt{\pi}}} \int_{-\infty}^{\infty} e^{-\left(\frac{r^2}{2\sigma^2}\right)} e^{-ikr}\, dr \qquad (23)$$
$$= \frac{e^{-\sigma^2\frac{k^2}{2}}}{\sqrt{2\pi\sigma\sqrt{\pi}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(r+ik\sigma^2)^2} dr$$

The integration in equation (23), .i.e.,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(r+ik\sigma^2)^2} dr$$

can be solved by using complex integration and the theorem of residue

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(r+ik\sigma^2)^2} dr = \sqrt{2\pi}\sigma \qquad (24)$$

Using equation (24) in equation (23), we have

$$a(k) = \sqrt{\frac{\sigma}{\sqrt{\pi}}} e^{-\sigma^2\frac{k^2}{2}} \qquad (25)$$

The functions $\psi(r)$ and $a(k) = a(p)$ are exhibited graphically in Fig 2 and Fig 3.

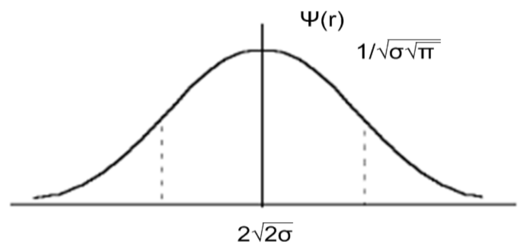

Figure 2:

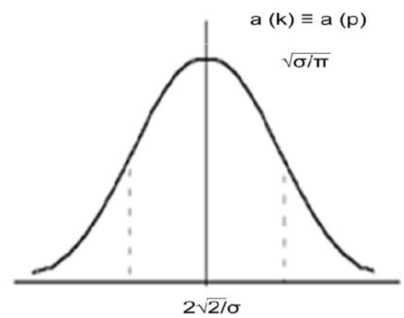

Figure 3.

Thus the Gaussian function $\psi(r)$ and Fourier transform $a(k) \equiv a(p)$are also Gaussian. Remember that the Gaussian function holds true for fractional Fourier transform (FRFT). Wavelets are apparently broken pieces of two superposing plane waves. The breaking up of superposing wave into waveletsis a time dependent phenomena. The curvature of time, .i.e., $\frac{\partial}{\partial t}$ in the same hyperbolic space is a manifestation of quantum optics. Wavelets of photons are quantized in the vicinity of light fringes,.i.e., in the same hyperbolic space (wiener space).

In a Young's double slit experiment, we deal the mathematical part with classical wave mechanics,.i.e., classical wave optics on the basis of optical path difference, the sizes of the slits, the separation in between the slits and the distance of the screen from the slits. We never give any heed to apparently broken wavelets in the vicinity of bright and dark fringes for both sides. This is where we lack the quantum mechanical measurements. Let the probability of waves with constructed interference be "1" (crust with crust) and destructive interference "0" (trough with trough). What is in between "1" and "0"?.We conjuncture that the wavelets (apparently broken) defines the probabilities of the wave functions with corresponding "fractional quantum states" ,.i.e., $(0.1 \leq n_f \leq 0.9)$. But these apparently broken wavelets are the manifestations of a continuous phenomena and represent transverse wavelet energies, may be, for photons, electrons and even for atoms. A wave packet or quanta with its corresponding quantum action, with an impact on slit, leads to diffraction which is a manifestation of quantum mechanical momentum,.i.e., $p = \hbar k$ where k is the propagation vector. The momentum of quanta is "wave impact" usually interpreted by photo electric and Compton effects, respectively. Therefore, wave particle duality for Young's double slit experiments is to be revisited. With force field after diffraction,.i.e., $F = \frac{dp}{dt} = \hbar\frac{dk}{dt}$, one can infer that k-space is changing with time. The changing k-space in time privides "frequencies of each of the wavelets different from each other and that the k-space is also inflated or stretched. Using Hamilton's principal that the index of refraction is the quantum mechanical analog of the classical momentum of the particle.

$$\lambda|\nabla_r p| \equiv \lambda\left|\frac{d\mu}{dt}\right| < 1, \left(k_q = \frac{\sqrt{2m(E-V)}}{\hbar}, k_{q,free} = \frac{\sqrt{2mE}}{\hbar}, \lambda = \frac{1}{k}, c = \lambda v\right)$$

With congruency, $\frac{d\mu}{dt}$ defines non locality for any dispersive medium, whereas $\nabla_r p \equiv |\nabla_r \mu|$ is the locality. The above expression shows that there is quantum entanglement both for dispersive and non-dispersive medium especially for wavelets. The eigen function of these wavelets are integrated with each other and we shall have a superposition of fractional quantum states in between (1,0) and (0,1) with frequencies different from each other and can be extended with a braid in a curvature of time even in non-inertial frame of references. For a non-dispersive medium, the curvature of time which is damped with space can represent the non-locality at a distance for beyond our imagination and hence the quantum entanglement in the Wigner space (reciprocal image). Wigner space follows inversion symmetry which can be extended for quantum entanglement to fix distance, may be, in the extra-terrestrial space, of course with its non-locality. Non-locality defines "invariance" of the system or space and follows Riemannian geometry. We shall take the project to apply Parsevals formulas later for the superposition of fractional quantum states as a linear combination of eigen function for the wavelets with corresponding commutating complex coefficients, of course with quantum entanglement. The quantum states 1,0 and $0.1 \leq nf \leq 0.9$are all integrated with a braid or wire. The quantum states 1 and 0follows the resonant states where as $0.1 \leq nf \leq 0.9$ follows the over damped and under damped states, respectively as logarithmic profiles. How the wave vector is changing with time? It is crystal clear with phase and group velocities, respectively. The phase velocity$v_p = \frac{E}{\hbar k} = \frac{E}{\mu}(p \equiv \mu)$ and the group velocity $v_g = \frac{dE}{\hbar\, dk}$ is considered in the Riemannian geometry. We shall yield symmetry operators for quantum entanglement. Remember that the Dirac Bosons precess (spin orbit coupling is negligible due to gyro scopic behavior)whereasDirac Fermions spin (spin orbit coupling is strong). Dirac Bosons are held at "knots" of the braid and precess to producing characteristic magnetic excitations. The Mexican hat wavelets are considered in the over damped conditions,.i.e., $0.1 \leq nf \leq 0.3$ and the Morlet wavelets in the under damped conditions,.i.e., $0.4 \leq nf \leq 0.9$. We consider the empty set comprising of fractional states between"1" and "0". With superposition of fractional states for Mexican hat and Morlet wavelet, knots are produced in the braid. These knots are invariant systems distributed on a braid or wire of quanta. Such behaviour will be dealt with Parsevals theorem, commutating complex coefficients for normalization, complex integration, residues and poles, the fractional charge quantization, stretching and eigen values.Considering the wave function (Eigen function) for a free particle that is for wavelets (apparently broken pieces of wave fronts of photons) .i.e., Schrodinger equation. Before we embark on wave function, writing Schrodinger equation.

$$H_{op}\psi(r,t) = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V(r,t)\right\}\psi(r,t) = -\frac{\hbar}{i}\frac{\partial}{\partial t}\psi(r,t) = E_{op}\psi(r,t) \quad (26)$$

and

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,t) + \frac{\hbar}{i}\frac{\partial}{\partial t}\psi(r,t) = -\frac{\hbar^2}{2m}\nabla^2\psi(r,t) - i\hbar\frac{\partial}{\partial t}\psi(r,t) = 0 \quad (27)$$

$$\psi(r,t) = \psi(r,0)e^{-\frac{i}{\hbar}Et} \quad (28)$$

Choosing a suitable quantum plane wave equation.i.e.,

$$\psi_q(r) = e^{\left[\pm\frac{i}{\hbar}(p.r - E.t)\right]} \quad (29)$$

where

$$\omega = \frac{E}{\hbar} = \frac{p^2}{2m\,\hbar}$$

The momentum and frequency are related through law of dispersion $E = \frac{p^2}{2m}$

$$\omega_p = \text{phase velocity} = \frac{\omega}{\hbar} = \frac{\hbar\omega}{k\hbar} = \frac{2\pi\hbar v}{\hbar k} = \frac{\hbar v}{p_{qm}} = \frac{E}{p_{qm}} \quad (30)$$

$$v_g = \text{group velocity} = \frac{d\omega}{dk} = \frac{d\hbar\omega}{dk\hbar} = \frac{d2\pi\hbar v}{dp_{qm}} = \frac{dE}{\hbar dk} \quad (31)$$

The plot of E versus k gives dispersion relationship for discrete and distinct quantize states. The superposition of a set of such waves defined by equation (29) for coupled quanta of photon can be written with an Eigen function in the momentum space as:

$$\psi(r,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} a(p)e^{-\frac{i}{\hbar}(p.r - E.t)}dp \quad (32)$$

The amplitude of component of momentum p is given by the Fourier transform

$$a(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \psi(r,t)e^{-\frac{i}{\hbar}(p.r - E.t)}dp \quad (33)$$

For convenience, we consider position momentum and energy as classical dynamic variables instead of their corresponding operators defined in equation (21). Rewriting equation (32) and by using equation(31) we have

$$\psi(r,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} a(p) e^{-\frac{i}{\hbar}\left[p.r-\left(\frac{p^2}{2m}\right)t\right]} dp \quad (34)$$

The width of the coupled quanta of photon quantum entanglement following the Gaussian distribution can be approximated

$$(\Delta r)^2 = < r^2 > - < r >^2 \quad (35)$$

Its origin is chosen at the centre of the coupled quanta of photons for a wave packet $s(t) = 0, < r >= 0$

$$(\Delta r)^2 = < r^2 > = \frac{1}{\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} r^2 e^{-\left(\frac{r^2}{\sigma^2}\right)} dr \quad (36)$$

But we know that

$$\int_{-\infty}^{\infty} r^{2\mathrm{n}-r^2}\, dr = \frac{1.3.5 \ldots (2n-1)\sqrt{\pi}}{2^n} (\text{factorial Functional integral}) (37)$$

$$(\Delta r)^2 = < r^2 > = \frac{1}{\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} r^2 e^{-\left(\frac{r^2}{\sigma^2}\right)} dr \quad (38)$$

$$\Delta r = \frac{\sigma}{\sqrt{2}} \quad (39)$$

Since we are dealing the coupled quanta of photon in a wave packet with Gaussian distribution, the uncertainty principle will require a minimum value equation (39)

$$< \Delta r >< \Delta p > \approx \frac{\hbar}{2} \quad (40)$$

Using equation (39) in equation (40) we have

$$< \Delta p > \approx \frac{\hbar}{2 < \Delta r >} \approx \frac{\hbar}{2\, \frac{\sigma}{\sqrt{2}}} = \frac{\hbar}{\sqrt{2}\sigma} \quad (41)$$

What we find from our calculation that $a(p)$ does not depend on time, t. The value of $< \Delta(p) >$ remains constant due to wave packet of coupled photon in a wave packet (defined by wave front). Thus,

$$\frac{\partial}{\partial r} = 0, \quad p_{op} = -i\hbar\nabla = -i\hbar\frac{\partial}{\partial r} = 0 \quad (42)$$

With superposition of wave, the wave packet of coupled photon is broken into wavelet thus the Gaussian distribution is changing with time. The Fourier transform (FT) of $\psi(r,0)$ is

$$a(p) = \frac{1}{\sqrt{2\pi\sigma\hbar\sqrt{\pi}}} \int_{-\infty}^{\infty} e^{-\left(\frac{r^2}{2\sigma^2}\right)-\left(\frac{i}{\hbar}\right)(p-<p>)^2\, r} dr \quad (43)$$

Using complex integration and theory of residue, the exponent in the integrand by combining and completing the square in r, we have

$$a(p) \equiv a(k_q) = \sqrt{\frac{\sigma}{\hbar\sqrt{\pi}}} e^{-\left(\frac{\sigma^2}{2\hbar^2}\right)(p-<p>)^2} \quad (44)$$

Equation (43) and (44) are obtained by using

$$\int f(z)dz = 2\pi i, \int_{-\infty}^{\infty} \frac{\sin z}{z} dz = \pi\, a(k) = 0 \int e^{-\frac{1}{2\sigma^2}\left(z+ik\sigma^2\right)^2}$$

$$= \int_{-\infty-ik\sigma^2}^{\infty-ik\sigma^2} e^{-\frac{1}{2\sigma^2}\left(z+ik\sigma^2\right)^2} = \sqrt{2\pi}\sigma$$

Equation (44) shows that the wave function sian in the momentum space. Putting equation (44) in equation (22), we have

$$\psi(r,t) = \sqrt{\frac{\sigma}{2\pi\hbar\sqrt{\pi}}} \int_{-\infty}^{\infty} e^{\left(\frac{\sigma^2}{2\hbar^2}\right)(p-<p>)^2} e^{\frac{i}{\hbar}\left[p.r-\left(\frac{p^2}{2m}\right)t\right]} dp \quad (45)$$

The integrand appearing in equation (45) is of the same form as that encountered in equation (43). We arrive at

$$\psi(r,t) = \frac{1}{\sqrt{\sigma\sqrt{\pi}}} \frac{\sigma}{\alpha} e^{-\left[\frac{r-\left(\frac{<p>}{m}\right)t}{2\alpha^2}\right]^2} e^{\left[\frac{i}{\hbar}<p>\left(r-\frac{<p>}{2m}t\right)\right]} \quad (46)$$

Where

$$\alpha^2 = \sigma^2 + \frac{i\hbar t}{m} \quad (47)$$

It is now evident that the wave packet travels with group velocity $V_g = \frac{<p>}{m}$ but $V_g = \frac{dE}{\hbar dk}$This is an indication that it follows dispersion relationship of quantum physics. The index of refraction is the quantum mechanical analogue of the classical momentum of the particle. The second exponent in equation (46) shows that the velocity V_p in quantum physics is $\frac{E}{\hbar k} = \frac{E}{p}$ which is related to half of the group velocity of photon in a wave packet. The probability density or intensity of bright fringes is obtained as the absolute square of ψ,.i.e.,

$$|\psi(r,t)|^2 = \frac{\sigma}{|\alpha|^2\sqrt{\pi}} e^{-\frac{\sigma^2\left[r-\left(\frac{<p>}{m}\right)t\right]^2}{|\alpha^4|}} \quad (48)$$

Hence width of the packet at time t, is

$$\Delta r = \frac{\sigma}{\sqrt{2}} \sqrt{1 + \frac{\hbar^2 t^2}{\sigma^4 m^2}} \quad (49)$$

Where $|\psi|^2$ has the form $e^{-\left(\frac{r^2}{\sigma^2}\right)}$ then $\Delta x = \frac{\sigma}{\sqrt{2}}$ we now show that

$$\frac{\Delta r(t)}{\Delta r(0)} = \frac{\sqrt{2}\Delta r}{\sigma}$$

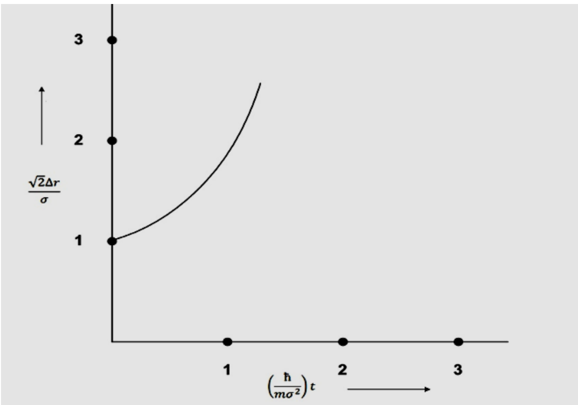

Figure 4: dilated Gaussian Profile with time dilation

The effect of dispersion is to spread out the Gaussian wave packet as a function of time. With expansion of Gaussian wave packet for coupled photons, the wave packet is broken into wavelets. These wavelets are distributed discretely and distinctively on the dilated Gaussian profile. We find that time dilation is not a realistic phenomena, but quantum behaviour with quantization of events .i.e., $\hbar p$. This happens when space is expanding. With space shrinking in a non inertial frame of reference time dilation is evident and is a manifestation of special theory of relativity. This is subject to postulate that the speed of light is constant.

In a Young's double slit experiment, the quantum optics poses the condition that the behavior of duality "quanta" can be interpreted as a "metaphor" due to quantum entanglement. With this metaphorical state, simultaneous duality, both particle and wave, especially for a "free quanta" is existing. Every matter or sub matter particle can be envisaged in a metaphorical state. The quantized matter or sub matter particles, the electrons or photons, in our case, gain extra energy from virtual particles due to exchange field may befroHigg's field as harmonic and adiabatic perturbation. Wavelets in the form of Mexican hat and Morlet are produced in between constructive ("1" states) and destructive ("0" states) pattern with itself a part of that, the states are connected with each other in the form of fractional quantum states $(0.1 \leq nf \leq 0.9)$. The fractional quantum states, for a "free quanta" for electrons and protons are relatively weak and travel in the Wigner space, where for quantized matter or sub matter, it takes the shape of a quantum braid wire in the invisible region (ultra violet) where fractional states gain energy from virtual particle (like the Higgs field). The Mexican hat wavelet are the manifestation of over damped oscillators $(0.1 \leq nf \leq 0.3)$ and Morlet wavelets of under damped oscillators $(0.4 \leq nf \leq 0.9)$ with perturbation of Higg's field for quantized electron and protons. For super position of quantum bit's "1" and "0"we have four significant states $(1,1), (1,0), (0,1), (0,0)$ and so it is the case for fractional states $(0.1 \leq nf \leq 0.9)$,.i.e.,2^9 fractional significant quantum states$2^{nf} \equiv$,i.e., Hermite fraction.

The coupled photons (in our case with Mexican hat and Morlet wavelets) follow ' quantumentanglement'which is recenlly and indeed experimentally verified [10]. The non locality especially in a non-dispersive medium (empty space) leads to the existence of dark matter or dark energy with anti-matter particle like antiprotons and positrons. [11]. Thus, the locality of fractional states work for dispersive medium and indeed for a dispersive.

CONCLUSION

We infer the following conclusion from the present study:
The increase in the length of quanta, i.e., stretching of quanta is $r_{op} \equiv \Delta r = \frac{\sigma}{\sqrt{2}}\sqrt{1+\frac{\hbar^2 t^2}{\sigma^4 m^2}}$ (maintains $c = \lambda\nu$with adiabatic perturbation) the twisting angle is responsible to producing "quantization of events" in our case the Mexican hat and Morlet wavelets. The mathematical expression obtained is $\alpha^2 = \hbar t\sigma^2 + \frac{i\hbar t}{m}$ (maintain harmonic perturbation). Resonant states for superposition are produced with states "1" and "0" for light and dark fringes respectively. The wavelets (Mexican hat and Morlet) are produced between "1" and "0" with commutation of I and sets (operators) and vice versa. The Mexican hat wavelets produced in the over damped quantum states $(0.1 \leq nf \leq 0.3)$ with logarithmic profile) where as the Morlet wavelets in the under damped quantum states $(0.4 \leq nf \leq 0.9)$ with logarithmic profile). Such a configuration of quanta followed "1" and "0" we have from significant resonant states $(1,1), (1,0), (0,1), (0,0)$ and so in the case for fractional significant quantum states $(2^{nf} \equiv H_{nf}$ is Hermite function.

Energy profile represents energy profiles or its energy distribution. Energy distribution is then interpreted for its measurement with probabilities superposition of Mexican hat and Morlet wavelets will be dealt exactly in the same manner as for matter waves or photons by Parsevel's theorem with complex coefficients, complex integration, residues and poles, factorial function and in the quantum states of entanglement. The quantum states of entanglement is a manifestation of twisting angle and quantization of events. The quantization of events, such as interpreted by us with the velocity of radiations which follows non-locality.

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