EXISTENCE OF COUPLED FIXED POINTS USING GENERALIZED (ψ,β)-ORDER CONTRACTIVE MAPPINGS

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ABSTRACT. *Gordji et al. [M. E. Gordji, M. Ramezani, Y. J. Cho and S. Pirbavafa, A generalization of Geraghty's theorem in partially ordered metric spaces and applications to ordinary differential equations, Fixed Point Theory Appl., 2012, 2012:74] obtained a generalization of Geraghty's theorem. In this paper, we introduce generalized (ψ,β)-order contractive mappings and obtain existence of coupled fixed point of such mappings in partially ordered metric spaces. These results establish some of the most general coupled fixed point theorems in partially ordered metric spaces. We also present an example and an application to demonstrate the results presented herein.*

Key Words: Coupled fixed point, mixed monotone property, well-ordered, partially ordered set.

1. INTRODUCTION

Fixed point theory is one of the well known traditional theories in mathematics that has a broad set of applications. Existence of fixed points of mappings satisfying certain contractive conditions can be employed to prove existence of solution of many functional equations (see for example [2, 6-8, 10, 13, 16, 21] and references mentioned therein). Recently, results related to existence and uniqueness of fixed points in complete metric spaces equipped with a partial ordering $\leq$ have been studied extensively in [1, 3-5, 9, 11, 12, 19]. Most of them deal with a monotone (either order-preserving or order-reversing) mappings, a classical contractive conditions and an assumption of existence of a lower or upper fixed point of mapping involved therein. The first result in this direction was established by Ran and Reurings [20] who also presented applications to linear and nonlinear metric space. Subsequently, Nieto and Rodriguez-Lopez [15] extended the result of Ran and Reurings [20] for non-decreasing mappings and applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions.

Bhashkar and Lakshmikantham [7] introduced a concept of coupled fixed point of a mapping and investigated some coupled fixed point results in partially ordered complete metric spaces. As an application of their result, they investigated the existence and uniqueness of solution for a periodic boundary value problem. Choudhury and Kundu [9] obtained coupled coincidence point results in partially ordered metric spaces for compatible mappings. Samet [18] proved coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. Then, later [3, 14, 16, 17, 19] provided some more interesting results in this direction.

In this paper, we study sufficient conditions for existence of a unique coupled fixed point of mapping satisfying generalized (ψ,β)-contractive condition defined on a partially ordered metric space. Presented results extend and unify various comparable results in existing literature. We also employ our result to establish an existence of solution of implicit integral equation.

In the sequel, □, □$^+$ and □ denote the set of real numbers, the set of nonnegative integers and the set of positive integers, respectively. The usual order on □ (respectively, on □$^+$) will be indistinctly denoted by $\leq$ or by $\geq$.

The following definitions and results will be needed in the sequel.

Let S be the class of all mappings $\beta:$ □$^+ \to [0,\frac{1}{2})$ satisfying the condition $\beta(t_n)\to\frac{1}{2}$ whenever $t_n\to 0$. Note that $S\neq\varnothing$ as the mapping $f:[0,\infty)\to[0,\frac{1}{2})$ given by the formula $f(x)=\frac{1}{2+x^2}$ qualifies for a member of S.

Define $\Psi=\{\psi:$ □$^+\to$ □$^+$: ψ is continuous, non-decreasing, subadditive and $\psi(t)=0$ iff $t=0\}$.

Definition 1.1. *Let X be a nonempty set. The triple $(X,\prec,d)$ is called a partially ordered metric space iff:*

(i) $\prec$ is a partial order on X ;

(ii) d is a metric on X .

Recall that if $(X,\prec)$ is a partially ordered set and $f:X\to X$ is such that for $x,y\in X$, $x\prec y$ implies

$f(x) \prec f(y)$, then f is said to be non-decreasing. Similarly, a non-increasing mapping is defined.

Definition 1.2. *A partially ordered metric space* $(X, \prec, d)$ *is said to have a sequential limit comparison property if every non-decreasing sequence (non-increasing sequence)* $\{x_n\}$ *in* X *such that* x_n *converges to* x *implies that* $x_n \prec x \, (x \prec x_n)$.

Let $(X, \prec)$ be a partially ordered set and $F : X \times X \to X$. According to [BhashkarL], if F is monotone non-decreasing in x and monotone non-increasing in y, then F is said to have *mixed monotone property*, that is, for any $x, y \in X$,

$$x_1,\ x_2 \in X,\ x_1 \prec x_2 \text{ implies } F(x_1, y) \prec F(x_2, y)$$

and

$$y_1,\ y_2 \in X,\ y_1 \prec y_2 \text{ implies } F(x, y_1) \succ F(x, y_2).$$

Let X be a nonempty set and $F : X \times X \to X$ be a mapping. A point (x, y) in $X \times X$ is called *coupled fixed point* of F if $F(x, y) = x$ and $F(y, x) = y$.

In this paper, we shall consider the set $\Delta_{\prec} \subseteq X \times X$ as follows:

$$\Delta_{\prec} = \{(x, y), (u, v) \in X \times X \ : \ x \prec u \text{ and } y \succ v\}.$$

2. The Coupled fixed point result

Now, we prove the existence of coupled fixed point of a mapping satisfying generalized (ψ, β)-contractive condition in the setup of partially ordered metric spaces.

Theorem 2.1. *Let* $(X, \prec, d)$ *be a complete partially ordered metric space and* $F : X \times X \to X$ *a mapping having mixed monotone property. Suppose that there exist* $\beta \in S$ *and* $\psi \in \Psi$ *such that*

$$\psi(d(F(x, y), F(u, v))) \le \beta(M(x, y, u, v)) N(x, y, u, v), \quad (2.1)$$

is satisfied for all $x, y, u, v \in X$, *for which* $x \prec u$ *and* $y \succ v$, *where*

$$M(x, y, u, v) = \psi\Bigg(\max\Big\{\frac{d(x, u) + d(y, v)}{2}, \frac{d(F(x, y), x) + d(F(x, y), u)}{2}, \frac{d(y, v) + d(F(x, y), x)}{2}, \frac{d(y, v) + d(F(x, y), u)}{2}\Big\}\Bigg)$$

and

$$N(x, y, u, v) = \psi\Bigg(\max\Big\{\frac{d(x, u) + d(y, v)}{2}, \frac{d(y, v) + d(F(x, y), x)}{2}\Big\}\Bigg).$$

If there exist x_0 *and* y_0 *in* X *such that* $x_0 \prec F(x_0, y_0)$ *and* $y_0 \succ F(y_0, x_0)$, *and either* F *is continuous or* X *has a sequential limit comparison property, then* F *has a coupled fixed point. Moreover,* F *has a unique coupled fixed point if and only if the set of all coupled fixed points is contained in* $\Delta_{\prec}$.

Proof: Let $x_0, y_0 \in X$ be such that $x_0 \prec F(x_0, y_0)$ and $y_0 \succ F(y_0, x_0)$. Construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n), \quad \text{for all } n \ge 0.$$

We shall show that $x_n \prec x_{n+1}$ and $y_n \succ y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Note that $x_0 \prec F(x_0, y_0) = x_1$ and $y_0 \succ F(y_0, x_0) = y_1$. That is, $x_0 \prec x_1$ and $y_0 \succ y_1$. So our claim is true for $n = 0$. Suppose that it holds for some fixed $n = k > 0$, that is, $x_k \prec x_{k+1}$ and $y_k \succ y_{k+1}$. Now we show that $x_{k+1} \prec x_{k+2}$ and $y_{k+1} \succ y_{k+2}$. Since F has mixed monotone property, therefore

$$x_{k+2} = F(x_{k+1}, y_{k+1}) \succ F(x_k, y_{k+1}) \succ F(x_k, y_k) = x_{k+1},$$

and

$$y_{k+2} = F(y_{k+1}, x_{k+1}) \prec F(y_k, x_{k+1}) \prec F(y_k, x_k) = y_{k+1}.$$

Hence the claim follows. By induction, we obtain that $x_{n+1} \prec x_{n+2}$ and $y_{n+1} \succ y_{n+2}$ for all $n \in \mathbb{N} \cup \{0\}$. Now from (2.1), we have

$$\psi(d(x_n,x_{n+1}))=\psi(d(F(x_{n-1},y_{n-1}),F(x_n,y_n)))$$
$$\leq\beta(M(x_{n-1},y_{n-1},x_n,y_n))$$
$$N(x_{n-1},y_{n-1},x_n,y_n), \quad (2.2)$$

where

$$M(x_{n-1},y_{n-1},x_n,y_n)$$
$$=\psi\Bigg(\max\Bigg\{\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2},$$
$$\frac{d(F(x_{n-1},y_{n-1}),x_{n-1})+d(F(x_{n-1},y_{n-1}),x_n)}{2},$$
$$\frac{d(y_{n-1},y_n)+d(F(x_{n-1},y_{n-1}),x_{n-1})}{2}$$
$$,\frac{d(y_{n-1},y_n)+d(F(x_{n-1},y_{n-1}),x_n)}{2}\Bigg\}\Bigg)$$
$$=\psi\Bigg(\max\Bigg\{\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}$$
$$,\frac{d(x_n,x_{n-1})+d(x_n,x_n)}{2}$$
$$,\frac{d(y_{n-1},y_n)+d(x_n,x_{n-1})}{2}$$
$$,\frac{d(y_{n-1},y_n)+d(x_n,x_n)}{2}\Bigg\}\Bigg)$$
$$=\psi\Bigg(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\Bigg)$$

and

$$N(x_{n-1},y_{n-1},x_n,y_n)=\psi\Bigg(\max\Bigg\{\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2},$$
$$\frac{d(y_{n-1},y_n)+d(F(x_{n-1},y_{n-1}),x_{n-1})}{2}\Bigg\}\Bigg)$$
$$=\psi\Bigg(\max\Bigg\{\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2},$$
$$\frac{d(y_{n-1},y_n)+d(x_n,x_{n-1})}{2}\Bigg\}\Bigg)$$
$$=\psi\Bigg(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\Bigg).$$

Thus from (2.2), we have

$$\psi(d(x_n,x_{n+1}))$$
$$\leq\beta\Bigg(\psi\Bigg(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\Bigg)\Bigg)\psi\Bigg(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\Bigg).$$

Similarly

$$\psi(d(y_n,y_{n+1}))=\psi(d(F(y_{n-1},x_{n-1}),F(y_n,x_n)))$$
$$\leq\beta(M(y_{n-1},x_{n-1},y_n,x_n))N(y_{n-1},x_{n-1},y_n,x_n), \quad (2.4)$$

where

$$M(y_{n-1},x_{n-1},y_n,x_n)=\psi\Bigg(\max\Bigg\{\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2},$$
$$\frac{d(F(y_{n-1},x_{n-1}),y_{n-1})+d(F(y_{n-1},x_{n-1}),y_n)}{2},$$
$$\frac{d(x_{n-1},x_n)+d(F(y_{n-1},x_{n-1}),y_{n-1})}{2},$$
$$\frac{d(x_{n-1},x_n)+d(F(y_{n-1},x_{n-1}),y_n)}{2}\Bigg\}\Bigg)$$
$$=\psi\Bigg(\max\Bigg\{\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2},\frac{d(y_n,y_{n-1})+d(y_n,y_n)}{2},$$
$$\frac{d(x_{n-1},x_n)+d(y_n,y_{n-1})}{2},\frac{d(x_{n-1},x_n)+d(y_n,y_n)}{2}\Bigg\}\Bigg)$$
$$=\psi(\max\{\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2},\frac{d(y_n,y_{n-1})}{2},\frac{d(x_{n-1},x_n)}{2}\}\Bigg)$$
$$=\psi\Bigg(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\Bigg)$$

and

$$N(y_{n-1},x_{n-1},y_n,x_n)=\psi\Bigg(\max\Bigg\{\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2},$$
$$\frac{d(x_{n-1},x_n)+d(F(y_{n-1},x_{n-1}),y_{n-1})}{2}\Bigg\}\Bigg)$$
$$=\psi\Bigg(\max\Bigg\{\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2},$$
$$\frac{d(x_{n-1},x_n)+d(y_n,y_{n-1})}{2}\Bigg\}\Bigg)$$
$$=\psi\Bigg(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\Bigg).$$

Thus from (2.4), we have

$$\psi(d(y_n,y_{n+1}))$$
$$\leq\beta\Bigg(\psi\Bigg(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\Bigg)\Bigg)\psi\Bigg(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\Bigg).$$

Adding (2.3) and (2.5), we get

$$\frac{\psi(d(x_n,x_{n+1}))+\psi(d(y_n,y_{n+1}))}{2}$$
$$\le \beta\left(\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right)\right)\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right).$$

As ψ is sub additive, so we have

$$\psi(d(x_n,x_{n+1})+d(y_n,y_{n+1}))$$
$$\le 2\beta\left(\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right)\right)\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right).$$

If we take $D_n = d(x_{n-1},x_n)+d(y_{n-1},y_n)$, then we have

$$\psi(D_{n+1}) \le 2\beta\left(\psi\left(\frac{D_n}{2}\right)\right)\psi\left(\frac{D_n}{2}\right), \qquad (2.6)$$

which shows that $\{D_n\}$ is a decreasing sequence.

We claim that $\lim_{n\to\infty} D_n = 0$.

If not, then there exits $r > 0$ such that $\lim_{n\to\infty} D_n = r$. Now from (tag2.7), we have

$$\frac{\psi(D_{n+1})}{\psi(\frac{D_n}{2})} \le 2\beta\left(\psi\left(\frac{D_n}{2}\right)\right),$$

which on taking limit as $n\to\infty$ gives that $\psi(r) < \psi(\frac{r}{2})$, a contradiction. So $r = 0$.

Now, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X . For this, we prove that $\lim_{n,m\to\infty}\psi(d(x_n,x_m)) = 0$ and $\lim_{n,m\to\infty}\psi(d(y_n,y_m)) = 0$, where $n,m\in\mathbb{N}$, with $n<m$.

Consider,

$$\begin{aligned}\psi(d(x_n,x_m)) &\le \psi(d(x_n,x_{n+1}))+\psi(d(x_{n+1},x_{m+1}))+\psi(d(x_m,x_{m+1}))\\ &\le \psi(d(x_n,x_{n+1}))+\psi(d(F(x_n,y_n),F(x_m,y_m)))\\ &\quad +\psi(d(x_m,x_{m+1}))\\ &\le \psi(d(x_n,x_{n+1}))+\beta(M(x_n,y_n,x_m,y_m))N(x_n,y_n,x_m,y_m)\\ &\quad +\psi(d(x_m,x_{m+1})),\end{aligned}$$

where

$$N(x_n,y_n,x_m,y_m) = \psi\left(\max\left\{\frac{d(x_n,x_m)+d(y_n,y_m)}{2},\frac{d(y_n,y_m)+d(F(x_n,y_n),x_n)}{2}\right\}\right)$$
$$= \psi\left(\max\left\{\frac{d(x_n,x_m)+d(y_n,y_m)}{2},\frac{d(y_n,y_m)+d(x_{n+1},x_n)}{2}\right\}\right).$$

Thus from (2.7), we have

$$\begin{aligned}\psi(d(x_n,x_m)) &\le \psi(d(x_n,x_{n+1}))+\beta(M(x_n,y_n,x_m,y_m))\\ &\quad +\psi\left(\max\left\{\frac{d(x_n,x_m)+d(y_n,y_m)}{2},\right.\right.\\ &\quad \left.\left.\frac{d(y_n,y_m)+d(x_{n+1},x_n)}{2}\right\}\right)+\psi(d(x_m,x_{m+1})).\end{aligned}$$

(2.8)

In a similar way, we obtain

$$\begin{aligned}\psi(d(y_n,y_m)) &\le \psi(d(y_n,y_{n+1}))+\beta(M(y_m,x_m,y_n,x_n))\\ &\quad +N(y_m,x_m,y_n,x_n)+\psi(d(y_m,y_{m+1}))\\ &\le \psi(d(y_n,y_{n+1}))+\beta(M(y_m,x_m,y_n,x_n))\\ &\quad +\psi\left(\max\left\{\frac{d(x_n,x_m)+d(y_n,y_m)}{2},\right.\right.\\ &\quad \left.\left.\frac{d(x_n,x_m)+d(y_{n+1},y_n)}{2}\right\}\right)+\psi(d(y_m,y_{m+1})).\end{aligned}$$

(2.9)

From (2.8) and (2.9), we have

$$\psi(d(x_n,x_m)+d(y_n,y_m))$$
$$\le \psi(d(x_n,x_{n+1}))+\beta(M(x_n,y_n,x_m,y_m))$$
$$\psi\left(\max\left\{\frac{d(x_n,x_m)+d(y_n,y_m)}{2},\frac{d(y_n,y_m)+d(x_{n+1},x_n)}{2}\right\}\right)$$
$$+\psi(d(x_m,x_{m+1}))+\psi(d(y_n,y_{n+1}))+\beta(M(y_m,x_m,y_n,x_n))$$
$$\psi\left(\max\left\{\frac{d(x_n,x_m)+d(y_n,y_m)}{2},\frac{d(x_n,x_m)+d(y_{n+1},y_n)}{2}\right\}\right)+\psi(d(y_m,y_{m+1}))$$
$$\le \psi(d(x_n,x_{n+1}))+\beta(M(x_n,y_n,x_m,y_m))\psi\left(\frac{d(y_n,y_m)}{2}\right)$$
$$+\beta(M(x_n,y_n,x_m,y_m))\psi\left(\frac{d(x_{n+1},x_n)}{2}\right)+\psi(d(x_m,x_{m+1}))$$
$$+\psi(d(y_n,y_{n+1})+\beta(M(y_m,x_m,y_n,x_n))\psi\left(\frac{d(x_n,x_m)}{2}\right)$$
$$+\beta(M(y_m,x_m,y_n,x_n))\psi\left(\frac{d(y_{n+1},y_n)}{2}\right)+\psi(d(y_m,y_{m+1}))$$

$$\begin{aligned}&\le \psi(d(x_n,x_{n+1}))+2\beta(M(x_n,y_n,x_m,y_m))[\psi(d(y_n,y_m)\\ &\quad +d(x_n,x_m))]+\beta(M(x_n,y_n,x_m,y_m))\psi\left(\frac{d(x_{n+1},x_n)}{2}\right)\\ &\quad +\psi(d(x_m,x_{m+1}))+\psi(d(y_n,y_{n+1}))\\ &\quad +\beta(M(y_m,x_m,y_n,x_n))\psi(\frac{d(y_{n+1},y_n)}{2})+\psi(d(y_m,y_{m+1})),\end{aligned}$$

which further implies that

$$\psi(d(x_n,x_m)+d(y_n,y_m))-2\beta(M(x_n,y_n,x_m,y_m))[\psi(d(y_n,y_m)+d(x_n,x_m))]$$
$$\leq\psi(d(x_n,x_{n+1}))+\beta(M(x_n,y_n,x_m,y_m))\psi\left(\frac{d(x_{n+1},x_n)}{2}\right)+\psi(d(x_m,x_{m+1}))$$
$$+\psi(d(y_n,y_{n+1}))+\beta(M(y_m,x_m,y_n,x_n))\psi\left(\frac{d(x_{n+1},x_n)}{2}\right)+\psi(d(y_m,y_{m+1})),$$

that is,

$$\psi(d(x_n,x_m)+d(y_n,y_m))<\frac{1}{1-2\beta(M(x_n,y_n,x_m,y_m))}[\psi(d(x_n,x_{n+1}))$$
$$+\psi\left(\frac{d(x_{n+1},x_n)}{2}\right)+\psi(d(x_m,x_{m+1}))$$
$$+\psi(d(y_n,y_{n+1}))+\psi\left(\frac{d(x_{n+1},x_n)}{2}\right)+\psi(d(y_m,y_{m+1}))]$$

which on taking limit as $m,n\to\infty$, gives

$$\lim_{n,m\to\infty}\psi(d(x_n,x_m)+d(y_n,y_m))\leq 0,$$

a contradiction. Hence $\lim_{n,m\to\infty}d(x_n,x_m)=0$ and $\lim_{n,m\to\infty}d(y_n,y_m)=0$, that is, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. As (X,d) is a complete, so there exist $x^*,y^*\in X$ such that $\lim_{n\to\infty}x_n=x^*$ and $\lim_{n\to\infty}y_n=y^*$.

Now we are to show that (x^*,y^*) is the coupled fixed point of F. If function F is continuous, then

$$d(F(x^*,y^*),x)\leq d(F(x^*,y^*),x_{n+1})+d(x_{n+1},x^*)$$
$$=d(F(x^*,y^*),F^n(x_0,y_0))+d(x_{n+1},x^*)$$
$$=d(F(x^*,y^*),F^n(x_0,y_0))+d(x_{n+1},x^*)$$
$$=d(F(x^*,y^*),F(F^{n-1}(x_0,y_0),F^{n-1}(y_0,x_0)))+d(x_{n+1},x^*)$$

on taking limit as $n\to\infty$ implies that $d(F(x^*,y^*),x^*)\leq 0$ and so $F\left(x^*,y^*\right)=x^*$. Similarly we have $F\left(y^*,x^*\right)=y^*$. Hence $\left(x^*,y^*\right)$ is the coupled fixed point of F. Suppose that F is not continuous, then by the sequential limit comparison property of X, we have $x_n\prec x^*$ and $y_n\succ y^*$. Now, from (2.1) we have

$$\psi(d(x_{n+1},F(x^*,y^*)))=\psi(d(F(x_n,y_n),F(x^*,y^*)))$$
$$\leq\beta(M(x_n,y_n,x^*,y^*))N(x_n,y_n,x^*,y^*), \tag{2.10}$$

where

$$N(x_n,y_n,x^*,y^*)=\psi\left(\max\left\{\frac{d(x_n,x^*)+d(y_n,y^*)}{2},\frac{d(y_n,y^*)+d(F(x_n,y_n),x_n)}{2}\right\}\right)$$
$$=\psi\left(\max\left\{\frac{d(x_n,x^*)+d(y_n,y^*)}{2},\frac{d(y_n,y^*)+d(x_{n+1},x_n)}{2}\right\}\right).$$

Thus form (2.10), we have

$$\psi(d(x_{n+1},F(x^*,y^*)))$$
$$\leq\beta(M(x_n,y_n,x^*,y^*))\psi\left(\max\left\{\frac{d(x_n,x^*)+d(y_n,y^*)}{2},\frac{d(y_n,y^*)+d(x_{n+1},x_n)}{2}\right\}\right).$$

which on taking limit as $n\to\infty$ implies that $\psi(d(x^*,F(x^*,y^*)))\leq 0.$ Hence we obtain $x^*=F(x^*,y^*).$ Similarly we obtain $y^*=F\left(y^*,x^*\right).$

Now, suppose that the set of coupled fixed points of F is contained in $\Delta_{\prec}$. We are to show that F has a unique coupled fixed point. On the contrary suppose that there exists (x',y') in $X\times X$ such that $F(x',y')=x'$ and $F(y',x')=y'$ with $x^*\neq x'$ and $y^*\neq y'$. From (tag2.1), we have

$$\psi(d(x^*,x'))=\psi(d(F(x^*,y^*),F(x',y')))$$
$$\leq\beta(M(x^*,y^*,x',y'))N(x^*,y^*,x',y'), \tag{2.11}$$

where

$$N(x^*,y^*,x',y')=\psi\left(\max\left\{\frac{d(x^*,x')+d(y^*,y')}{2},\frac{d(y^*,y')+d(F(x^*,y^*),x^*)}{2}\right\}\right)$$
$$=\psi\left(\max\left\{\frac{d(x^*,x')+d(y^*,y')}{2},\frac{d(y^*,y')}{2}\right\}\right)$$
$$=\psi\left(\frac{d(x^*,x')+d(y^*,y')}{2}\right).$$

From (2.11),

$$\psi(d(x^*,x'))\leq\beta(M(x^*,y^*,x',y'))\psi\left(\frac{d(x^*,x')+d(y^*,y')}{2}\right).$$

Similarly

$$\psi(d(y',y^*)) \le \beta(M(y',x',y^*,x^*))\psi\left(\frac{d(y',y^*)+d(x',x^*)}{2}\right).$$

Thus we have

$$\psi(d(x^*,x')+d(y',y^*)) \le [\beta(M(x^*,y^*,x',y')) + \beta(M(x^*,y^*,x',y'))]\psi\left(\frac{d(x^*,x')+d(y^*,y')}{2}\right) < \psi\left(\frac{d(y',y^*)+d(x',x^*)}{2}\right),$$

a contradiction. Hence $x^* = x'$ and $y^* = y'$.

Conversely, if F has a unique coupled fixed point then, the set of coupled fixed point is contained in $\Delta_{\prec}$.

Example 2.1. *Let* $X = [0,1]$ *be endowed with a usual order and a usual metric* d. *Consider the mapping*

$$F : X \times X \to X, \quad F(x,y) = \begin{cases} \frac{x-y}{8} & \text{if } x \ge y \\ 0 & \text{otherwise.} \end{cases}$$

As fo $x_1 \le x_2$ *, we have* $\frac{x_1-y}{8} \le \frac{x_2-y}{8}$ *for all* $y \in X$ *and also for* $y_1 \le y_2$*, we obtain* $\frac{x-y_1}{8} \ge \frac{x-y_2}{4}$ *for all* $x \in X$*, that is* F *has a mixed monotone property.*

Define β, ψ *as follows:* $\psi(x) = \ln(x+1)$ *and* $\beta(x) = \frac{\psi(x)}{2x}$. *Clearly,* $\beta \in S$ *and* $\psi \in \Psi$ *. Now, we show that* F *satisfies condition (2.1). For* $x \le u$ *and* $y \ge v$ *, we have*

$$\begin{aligned}
\psi(d(F(x,y),F(u,v))) &= \psi\left(d\left(\frac{x-y}{8},\frac{u-v}{8}\right)\right) \\
&= \psi\left(\frac{1}{8}|(x-u)-(y-v)|\right) \\
&\le \psi\left(\frac{1}{8}[|x-u|+|y-v|]\right) \\
&= \ln\left(\frac{1}{8}[|x-u|+|y-v|]+1\right) \\
&\le \frac{1}{2}\ln\left[\ln\left(\frac{1}{2}(|x-u|+|y-v|+1\right)+1\right] \\
&= \beta\left(\psi\left(\frac{1}{2}(|x-u|+|y-v|)\right)\right)\psi\left(\frac{1}{2}(|x-u|+|y-v|)\right) \\
&= \beta\left(\psi\left(\frac{d(x,u)+d(y,v)}{2}\right)\right)\psi\left(\frac{d(x,u)+d(y,v)}{2}\right) \\
&\le \beta(M(x,y,u,v))N(x,y,u,v).
\end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied. Moreover $(0,0)$ *is the unique coupled fixed point of* F .

Corollary 2.1. *Let* $(X,\prec,d)$ *be a complete partially ordered metric space and* $F : X \times X \to X$ *a mapping having mixed monotone property. Suppose that there exist* $\beta \in S$ *and* $\psi \in \Psi$ *such that*

$$\psi(d(F(x,y),F(u,v))) \le \beta\left(\psi\left(\frac{d(x,u)+d(y,v)}{2}\right)\right)\psi\left(\frac{d(x,u)+d(y,v)}{2}\right), \tag{2.12}$$

for all $x,y,u,v \in X$*, for which* $x \prec u$ *and* $y \succ v$*. If there exist* $x_0, y_0 \in X$ *such that* $x_0 \prec F(x_0,y_0)$ *and* $y_0 \succ F(y_0,x_0)$ *and either* F *is continuous or* X *has a sequential limit comparison property, then* F *has a coupled fixed point. Moreover,* F *has a unique coupled fixed point if and only if the set of all coupled fixed points is contained in* $\Delta_{\prec}$.

An application

Let $\Omega = [0,1]$ be a bounded set in □ , and $L^2(\Omega)$ a set of comparable functions on Ω whose square is integrable on Ω. Consider the integral equation

$$x(t) = \int_{\Omega} q(t,s,x(s))ds + k(t), \tag{3.1}$$

where $q:\Omega\times\Omega\times\mathbb{R}^+\to\mathbb{R}^+$ and $k:\Omega\to\mathbb{R}^+$ are continuous mappings.

In this section, we shall study sufficient condition for existence of solution of integral equation in the framework of ordered metric spaces.

Let $L^2(\Omega)$ be endowed with the partial ordered $\prec$ given by:

$x,y\in L^2(\Omega),\ x\prec y$ iff $x(t)\le y(t)$, for all $t\in\Omega$ and

$d:L^2(\Omega)\times L^2(\Omega)\to\mathbb{R}^+$ a metric given by

$$d(x,y)=\sup_{t\in\Omega}\left|x(t)-y(t)\right|.$$

Suppose that the following conditions holds:

1) For each $s,t\in\Omega,$

$$q(t,s,u(s))\le q\left(t,s,v(s)\right)$$

for all $u(s)\le v(s)$.

2) For each $s,t\in\Omega$

$$\int_\Omega\left|q(t,s,u(s))-q(t,s,v(s))\right|ds\le\left|u(s)-v(s)\right|.$$

3) There exist $\psi:[0,\infty)\to[0,\infty),$ continuous, non-decreasing with $\psi^{-1}(\{0\})=\{0\}$ and $\beta:[0,\infty)\to[0,\frac{1}{2})$ such that $\psi(3r/4)\le\beta(\psi(r))\psi(r)$ for all $r\in[0,\infty)$.

Then the integral equation (tag2.17) has a solution in $L^2(\Omega)$.

Proof. Define

$$F(x(t),y(t))=\int_\Omega q\left(t,s,\frac{3x(s)-y(s)}{8}\right)ds+k(t)$$

when $x(s)\ge y(s)$ and $F(x(t),y(t))=0$ otherwise. Now using $x_1(s)\le x_2(s)$ and (1), we have

$$\begin{aligned}F(x_1(t),y(t))&=\int_\Omega q\left(t,s,\frac{3x_1(s)-y(s)}{8}\right)ds+k(t)\\&\le\int_\Omega q\left(t,s,\frac{3x_2(s)-y(s)}{8}\right)ds+k(t)\\&=F(x_2(t),y(t)).\end{aligned}$$

Also, using $y_1(s)\ge y_2(s)$ and (1), we have

$$\begin{aligned}F(x(t),y_1(t))&=\int_\Omega q\left(t,s,\frac{3x(s)-y_1(s)}{8}\right)ds+k(t)\\&\le\int_\Omega q\left(t,s,\frac{3x(s)-y_2(s)}{8}\right)ds+k(t)\\&=F(x(t),y_2(t)).\end{aligned}$$

For $x_0(s)$, $y_0(s)$ belong to the Lebesgue space $L_2[0,1]$, we can choose the square integrable functions such that the two integrals of $q(t,s,\cdot)$ over $[0,1]$ is finite. By changing the values of $x_0(s)$ and $y_0(s)$ at one point(s), we have

$$x_0(s)\le\int_\Omega q\left(t,s,\frac{3x_0(s)-y_0(s)}{8}\right)ds+k(t),$$

and

$$y_0(s)\ge\int_\Omega q\left(t,s,\frac{3y_0(s)-x_0(s)}{8}\right)ds+k(t).$$

Thus, we have

$$\begin{aligned}x_0(s)&\le\int_\Omega q\left(t,s,\frac{3x_0(s)-y_0(s)}{8}\right)ds+k(t)\\&=F(x_0(t),y_0(t))\end{aligned}$$

and

$$\begin{aligned}y_0(s)&\ge\int_\Omega q\left(t,s,\frac{3y_0(s)-x_0(s)}{8}\right)ds+k(t)\\&=F(y_0(t),x_0(t)).\end{aligned}$$

Now, for $x(t)\le u(t)$ and $y(t)\ge v(t),$ we have

$$\psi(d(F(x,y),F(u,v)))$$
$$=\psi(\sup_{t\in\Omega}|F(x(t),y(t))-F(u(t),v(t))|)$$
$$=\psi\left(\sup_{t\in\Omega}\left|\begin{array}{l}\int_{\Omega}q(t,s,\frac{3x(s)-y(s)}{8})ds\\ -\int_{\Omega}q(t,s,\frac{3u(s)-v(s)}{8})ds\end{array}\right|\right)$$
$$\leq\psi\left(\sup_{t\in\Omega}\int_{\Omega}\left|\begin{array}{l}q(t,s,\frac{3x(s)-y(s)}{8})\\ -q(t,s,\frac{3u(s)-v(s)}{8})\end{array}\right|ds\right)$$
$$\leq\psi\left(\sup_{t\in\Omega}\left|\frac{3x(s)-y(s)}{8}-\frac{3u(s)-v(s)}{8}\right|\right)$$
$$=\psi\left(\sup_{t\in\Omega}\left|\frac{3[x(s)-u(s)]}{8}-\frac{[y(s)-v(s)]}{8}\right|\right)$$
$$\leq\beta\left(\psi\left(\frac{1}{2}[\sup_{t\in\Omega}|x(t)-u(t)|+\sup_{t\in\Omega}|y(t)-v(t)|]\right)\right)$$
$$\times\psi\left(\frac{1}{2}[\sup_{t\in\Omega}|x(t)-u(t)|+\sup_{t\in\Omega}|y(t)-v(t)|]\right)$$
$$=\beta(\psi(\frac{d(x,u)+d(y,v)}{2}))\psi(\frac{d(x,u)+d(y,v)}{2}).$$

Thus (2.12) is satisfied. Now we can apply Corollary 2.1 to obtain the solution of integral equation (3.1) in $L^2(\Omega)$.

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