GALERKIN'S FINITE ELEMENT FORMULATION FOR THIN FILM FLOW OF A THIRD GRADE FLUID DOWN AN INCLINED PLANE

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ABSTRACT. *In this paper, the governing non-linear equation of thin film flow problem with a third grade fluid on an inclined plane is solved and analyzed for velocity field using linear Lagrange polynomials in a Galerkin's finite element fashion. The approach having piecewise linear shape functions, provide better approximations than those produced by traditional perturbation technique as well as homotopy perturbation method. The numerical results state that the presented formulation is quite accurate and efficient for this kind of problems.*

Keywords**:** *Galerkin method; finite element method; third grade fluid, thin film flow.*

1. INTRODUCTION

In general most of the problems in science and engineering are nonlinear. Specifically, the governing flow problems of non-Newtonian fluids are highly nonlinear and have higher order than those of Navier-Stokes equations. The solutions of such flow problems are always a challenging task to mathematicians and computer programmers. The perturbations methods [1,2] have been widely applied in obtaining the solution of such flow problems. However, the perturbation solution required a small or large parameter into the equation and the solution is only valid for a very small range of the parameter values. To overcome these constraints some new methods based on the homotopy transformation are developed. Homotopy analysis method [3] is proposed by Liao as a generalization of the perturbation method which ensures the convergence of the developed series solution. He [4] blended the idea of homotopy and perturbation into homotopy perturbation method. Siddiqui et al. [5] used perturbation and homotopy perturbation methods for thin film flow of a third grade fluid down an inclined plane. The same problem was considered by Sajid et al. [6] and obtained a convergent series solution.

The purpose of the present investigation is to look for finite element solution for the thin film flow of a third grade fluid down an inclined plane. The Galerkin's finite element method (FEM) [7-13] based on weighted-residual formulation has been finding its applications in almost all branches of science and engineering. It has found applications in areas as diverse as solid mechanics, fluid dynamics, heat transfer and electromagnetism [9]. It is a well-established numerical technique in the field of solid mechanics [14]. Finite elements have been utilized in various different ways to solve boundary value problems. In some formulations a weighted residual approach is adopted, while in others variational approaches are considered.

In this paper, we set up finite element solution using linear Lagrange polynomials as the element and weight functions for the smooth solution of thin film flow of a third grade fluid down an inclined plane. Section 2 develops the equation governing the motion down an inclined plane. In Section 3 the Galerkin's finite element formulation is developed using linear Lagrange polynomial for the nonlinear governing differential equation. The graphical results are presented in section 4. Section 5 syntheses some concluding remarks.

2. Flow Equations

The thin film flow of a third order fluid down an inclined plane is governed by the boundary value problem [5]

$$\mu\frac{d^2u}{dx^2}+6(\beta_2+\beta_3)\left(\frac{du}{dx}\right)^2\frac{d^2u}{dx^2}+\rho g\sin\alpha=0, \quad (1)$$

$$u(0)=0, \quad \frac{du}{dx}=0 \text{ at } y=\delta, \quad (2)$$

where ρ the constant density, u the velocity along the inclined plane, μ is the dynamic viscosity, $\beta 2$ and $\beta 3$ are material constants of third grade fluid and δ is the thickness of the thin layer.

Defining

$$\bar{u}=\frac{u\delta}{\nu}, \quad \bar{y}=\frac{y}{\delta} \quad (3)$$

The problem in Eqs. (1) and (2) takes the form

$$\frac{d^2u}{dx^2}+6\beta\left(\frac{du}{dx}\right)^2\frac{d^2u}{dx^2}+\lambda=0, \quad (4)$$

$$u(0)=0, \quad \frac{du}{dx}=0 \text{ at } y=1. \quad (5)$$

in which

$$\beta=\frac{(\beta_2+\beta_3)\nu^2}{\mu\delta^4}, \quad \lambda=\frac{g\sin\alpha\delta^3}{\nu^2}.$$

In the next section, solution of the governing equation (4) subject to (5) by Galerkin's finite element method using linear Lagrange polynomials is presented.

2. Galerkin's Finite Element Formulation

The Galerkin's formulation [7-13] in the finite element environment requires that we choose a suitable trial or basis function that is applied locally over a typical finite element in the complete x domain. Let us denote this trial function by $\tilde{u}$. In this case it is necessary to satisfy inter-element compatibility with respect to displacements. In other words, the trial function is C^0-continuous. Each element has two nodes. We interpolate the function at each node of the element. This requires one unknown parameter at each node of the element.

$$\tilde{u} = a_1 + a_2 x \tag{6}$$

Rather than formulating the problem in terms of arbitrary constants a_1 & a_2, we prefer to recast the above linear trial function in terms of values of the dependent functions at nodes i & j (the convention used by Zienkiewicz and Stasa [7, 8]).

$$\tilde{u} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} \tag{7}$$

where $N_1 = x_j - x / x_j - x_i$ & $N_2 = x - x_i / x_j - x_i$, the trial function constants now are the nodal values of the dependent variable $\tilde{u}$, and N_i terms are the familiar interpolation, or shape, functions.

By using quasi linearization [15], governing differential equation takes the form:

$$u''_{n+1} - 12\left(\frac{\beta\rho g \sin\alpha}{(\mu + 6\beta u_n'^2)^2}\right) u'_n u'_{n+1} + 12\left(\frac{\beta\rho g \sin\alpha}{(\mu + 6\beta u_n'^2)^2}\right) u_n'^2 + \frac{\rho g \sin\alpha}{\mu + 6\beta u_n'^2} = 0 \tag{8}$$

We follow the standard Galerkin's approach and choose weighting function $w = N_i,\ i = 1, 2.$ Integrating over the entire region, we will get:

$$\int_X w \left(u''_{n+1} - 12\left(\frac{\beta\rho g \sin\alpha}{(\mu + 6\beta u_n'^2)^2}\right) u'_n u'_{n+1} + 12\left(\frac{\beta\rho g \sin\alpha}{(\mu + 6\beta u_n'^2)^2}\right) u_n'^2 + \frac{\rho g \sin\alpha}{\mu + 6\beta u_n'^2} \right) dx \tag{9}$$

where $'$ denotes differentiation with respect to y. For our particular problem, after substituting the trial functions into the equation (9), it can be written in discretized form as:

$$\sum_{e=1}^{n} \left(\int_X w' \tilde{u}'_{n+1} dx + 12 \int_X w \left(\frac{\beta\rho g \sin\alpha}{(\mu + 6\beta \tilde{u}_n'^2)^2} \right) \tilde{u}'_n \tilde{u}'_{n+1} dx - 12 \int_X w \left(\frac{\beta\rho g \sin\alpha}{(\mu + 6\beta \tilde{u}_n'^2)^2} \right) \tilde{u}_n'^2 dx - \int_X w \frac{\rho g \sin\alpha}{\mu + 6\beta \tilde{u}_n'^2} dx \right) = 0 \tag{10}$$

where $'e'$ represents the element and $'n'$ represents the total number of elements in the discretized region. In matrix notation, the system of equation (10) can be written as:

$$\sum_{e=1}^{n} \left((\mathbf{k_1} + \mathbf{k_2}) \mathbf{u} - \mathbf{f} \right) = \mathbf{0} \tag{11}$$

where

$$\mathbf{k_1} = \frac{1}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{12}$$

$$\mathbf{k_2} = \eta \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \tag{13}$$

$$\mathbf{f} = \left(\gamma\eta + \frac{\mu\lambda L^3}{2\xi} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{14}$$

where $\mathbf{k_1}$ and $\mathbf{k_2}$ are stiffness matrices and $\mathbf{f}$ is a force vector,

$\rho g \sin\alpha = \mu\lambda,\ \gamma = -u_i^n + u_j^n,\ \xi = \mu L^2 + 6\beta\gamma^2$ and $\eta = 6\beta\mu\lambda\gamma L^3 / \xi^2$.

By applying assembly procedure given in [7, 13], and using equation (11) for $'n'$ elements, we will get:

$$\left(\frac{1}{L} \begin{bmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix} + \eta \begin{bmatrix} -1 & 1 & & & & & \\ -1 & 0 & 1 & & & & \\ & -1 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots \\ & & & & -1 & 0 & 1 \\ & & & & & -1 & 1 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ \vdots \\ \vdots \\ u_n \\ u_{n+1} \end{bmatrix} = \left(\gamma\eta + \frac{\mu\lambda L^3}{2\xi} \right) \begin{bmatrix} 1 \\ 2 \\ 2 \\ \vdots \\ \vdots \\ \vdots \\ 2 \\ 1 \end{bmatrix} \tag{15}$$

Solving equation (15), we will get the results.

3. Numerical results and discussion Examples

In this section we have graphically given the velocity field obtained by the finite element method described in the previous section. The method is implemented using Matlab. The influence of parameters β and λ on the velocity u is displayed in Figs. 1 and 2. The effect of the third grade parameter β on the velocity field is shown in Fig. 1. Figure 1 shows that the velocity is a decreasing function of the third grade parameter. Figure 2 is made to show the variation of parameter λ on the velocity field. This Fig. elucidate that velocity increases by increasing the parameter λ. It is noted that the results obtained are quite comparable with those of [6].

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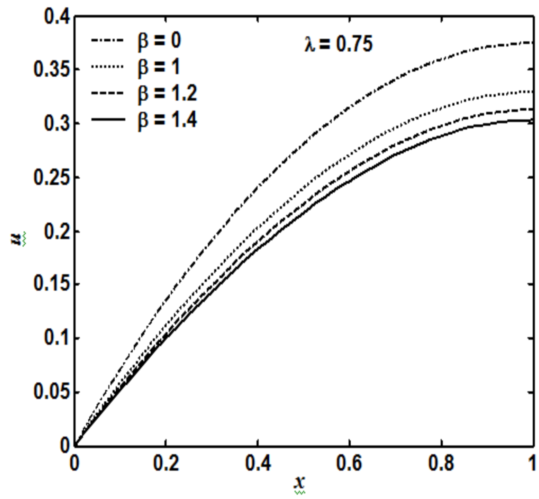

Fig. 1 Dimensionless velocity profiles with different values of β.

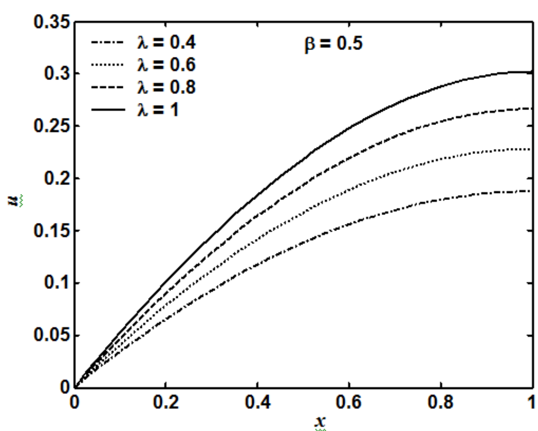

Fig. 2 Dimensionless velocity profiles with different values of λ .

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