ON SUPER EDGE-MAGIC TOTAL LABELING OF REFLEXIVE W-TREES[i]

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ABSTRACT. *Kotzig and Rosa [17] defined a magic labeling λ on a graph G to be a bijective mapping that assigns the integers from 1 to p+q to all the vertices and edges such that the sums of the labels on an edge and its two endpoints is constant for each edge. Ringel and Lladó [20] redefined this type of labeling as edge-magic. Recently, Enomoto et al. [6] introduced the name super edge-magic for magic labelings defined by Kotzig and Rosa, with an additional property that the vertices receive the smallest labels. That is,* $\lambda(V(G)) = \{1,2,3,...,p\}$.

If the domain of a labeling λ is the set of all vertices and edges of the graph G, then such labeling is called total labeling. The labelings which we study in this paper, have another property that, the weight $\omega(xy)$ $\forall xy \in E(G)$, *calculated as;* $\omega(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$, *is equal to a fixed constant k, called the magic constant or sometimes the valence of λ. A graph is called super edge-magic total (SEMT) if it admits a super edge-magic total labeling. In this paper, we construct new families of trees (using w-trees [15]), referred as reflexive w-trees and prove that they are super edge magic total. It is a classical problem to construct new classes of super edge magic total graphs using old ones.*

Keywords: super edge magic total labeling, w-trees, reflexive w-trees.

INTRODUCTION AND PRELIMINARY RESULTS

Let $G(V,E)$ be a finite, simple and undirected graph with $|V(G)| = p$ and $|E(G)| = q$. Kotzig and Rosa [17] defined a magic labeling λ on a graph G to be a bijective mapping that assigns the integers from 1 to $p+q$ to all the vertices and edges such that the sums of the labels on an edge and its two endpoints is constant for each edge. Ringel and Lladó [20] redefined this type of labeling as edge-magic. Recently, Enomoto et al. [6] introduced the name super edge-magic for magic labelings defined by Kotzig and Rosa, with an additional property that the vertices receive the smallest labels. That is, $\lambda(V(G)) = \{1,2,3,...,p\}$.

If the domain of a labeling λ is the set of all vertices and edges of the graph G, then such labeling is called total labeling. The labelings which we study in this paper, have another property that, the weight $\omega(xy)$ $\forall xy \in E(G)$, calculated as; $\omega(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$, is equal to a fixed constant k, called the magic constant or sometimes the valence of λ. A graph is called super edge-magic total (SEMT) if it admits a super edge-magic total labeling. Some labelings have only the vertex-set (edge-set) as their domain, such labelings are called vertex-labelings (edge-labelings). Other domains for the labeling λ are also possible. There are many types of graph labelings, for example, harmonius, cordial, graceful, equitable, multilevel, distance, antimagic, etc.

A number of classification problems on SEMT labelings of connected graphs have been investigated. Figueroa-Centeno et al. [8] proved the following:

- If G is a bipartite or tripartite (super) edge-magic graph, then nG is (super) edge-magic when n is odd.
- If m is a multiple of $n + 1$, then $K_{1,m} \cup K_{1,n}$ is super edge-magic.
- $K_{1,2} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 3.
- $P_m \cup K_{1,n}$ is super edge-magic when $m \geq 4$.
- $2P_n$ is super edge-magic if and only if n is not 2 or 3.
- $K_{1,m} \cup 2nK_2$ is super edge-magic for all m and n.

Figueroa-Centeno et al. [11] conjectured that $C_m \cup C_n$ is super edge-magic if and only if $m+n \geq 9$ and $m+n$ is odd. Baskoro and Ngurah [5] showed that nP_3 is super edge-magic for $n \geq 4$ and n even. Lee and Kong [18] use the notation $St(a_1, a_2, ..., a_n)$ to denote the disjoint union of the n stars $St(a_1)$, $St(a_2)$, ..., $St(a_n)$ of order a_1+1, a_2+1, ..., a_n+1, respectively. They proved the following graphs to be super edge-magic:

- $St(m,n)$ where $n \equiv 0 \mod (m + 1)$.
- $St(1,1,n)$, $St(1,2,n)$, $St(1,n,n)$, $St(2,2,n)$, $St(2,3,n)$, $St(1,1,2,n)$ for $n \geq 2$.

Lee and Kong in the same paper, conjectured that $St(a_1, a_2, ..., a_n)$ is super edge-magic when $n (> 1)$ is odd. It is known that if a graph $G(p,q)$ is super edge-magic, then $q \leq 2p - 3$ [6]. This bound can be improved for bipartite graphs of order $p \geq 4$, to be $q \leq 2p-5$ [19]. For more results concerning edge-magic total labelings, see [4,9,10] and a complete Gallian's survey paper [12]. Javaid et al. [15] studied the SEMT labeling of w-graph ($W(n)$) and w-tree ($WT(n,k)$). They defined a w-graph $W(n)$ to be a graph obtained by amalgamating a vertex from two stars $2St(n + 3)$.

In the proofs of the upcoming theorems, the summation of type $\sum_{i=1}^{0} f_i$ will be considered as 0. The theorems in this paper are proved using the lemma proposed by Figueroa et al. [7], stated below.

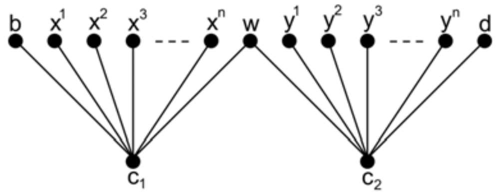

Fig. 1. W(n)

Lemma 1. A *(p,q)* graph *G* is super edge-magic total if and only if there exists a bijection $\lambda : V(G) \to \{1,2,\cdots,p\}$ such that the set of edge weights $S = \{\lambda(x) + \lambda(y) \mid xy \in E(G)\}$ consists of *q* consecutive integers. In such a case, λ extends to a super edge-magic total labeling of *G* with magic constant $k = p + q + s$, where $s = min(S)$ and
$S = \{k - (p + 1), k - (p + 2), k - (p + 3),..., k - (p + q)\}$.

2 MAIN RESULTS

In this section, we will study *super edge magic total* (SEMT) *labeling* of reflexive w-trees which are constructed from the generalized w-graphs defined below.

Definition 1. A *generalized w-graph* $W(n_{1,j}, r)$, for $1 \le j \le r$ and $r \in Z^+$, is a graph obtained from a sequence of r stars, each of order $n_{1,j} + 2$, by amalgamating a single vertex of the j^{th}-star with a vertex of the $(j+1)^{st}$-star, for $1 \le j \le r-1$.

Definition 2. A reflexive w-graph $RW(m; n_{i,j}; r_i)$, for $1 \le i \le 2$ and $1 \le j \le r_i$, is a graph obtained by joining two copies of generalized w-graph $W(n_{1,j}, r)$, with a path $y_1, y_2, \ldots, y_{m-2}$, by adding the edges $c_{11}y_1$ and $c_{21}y_{m-2}$ (see Figure 2).

In the first theorem, we prove that the *reflexive w-graphs* admit *super edge magic total labelig*.

Theorem 1. The graph $G \cong RW(m; n_{i,j}; r_i)$, for $1 \le i \le 2$, $1 \le j \le r_i$ and $m, n_{i,j}, r_i \in \mathbb{Z}^+$, is *super edge-magic total*.

Proof. The graph $G \cong RW(m; n_{i,j}; r_i)$, for $1 \le i \le 2$, $1 \le j \le r_i$, is of order $\sum_{i=1}^{2}\sum_{j=1}^{r_i} n_{ij} + 2(r_1 + r_2) + m,$ and size $\sum_{i=1}^{2}\sum_{j=1}^{r_i} n_{ij} + 2(r_1 + r_2) + m - 1.$ The vertex set of $RW(m; n_{i,j}; r_i)$ is defined as:

$V(G) = \{b_{i,j} : 1 \le i \le 2, 1 \le j \le r_i + 1\} \cup \{c_{i,j} :$
$1 \le i \le 2, 1 \le j \le r_i\}$
$\cup \{x^l_{i,t} : 1 \le l \le 2, 1 \le i \le r_l, 1 \le t \le n_{li}\}$
$\cup \{y_i : 1 \le i \le m-2\}$.

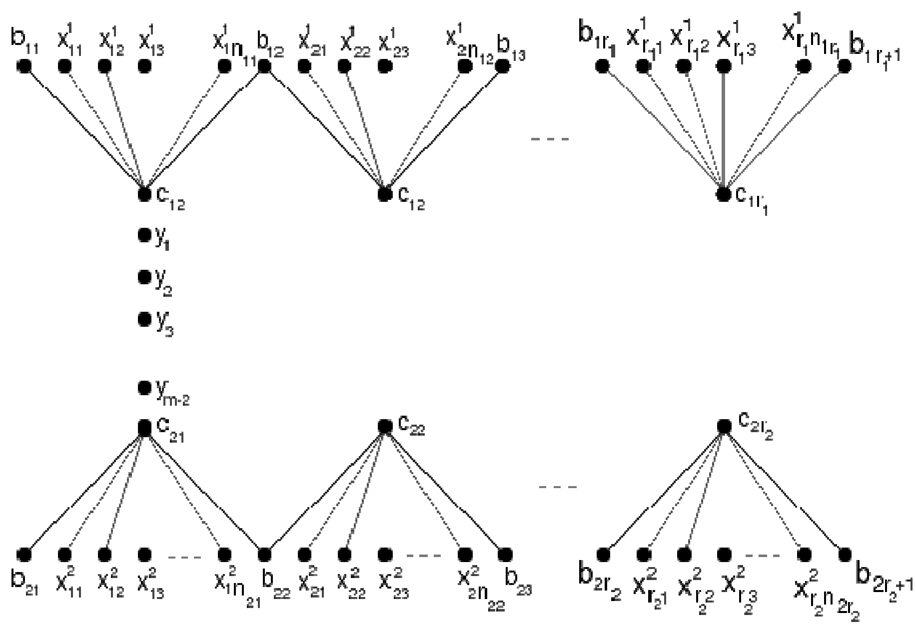

Figure 2: $RW(m; n_{i,j}; r_i)$, for $1 \le i \le 2$, $1 \le j \le r_i$

The edge set of G is defined as:

$$E(G) = \{y_1c_{11}, y_{m-2}c_{21}\} \cup \{x_{i,t}^l c_{l,i} : 1 \le l \le 2, 1 \le i \le r_l, 1 \le t \le n_{li}\} \cup \{b_{i,j}c_{i,j}, b_{i,j+1}c_{i,j} : 1 \le i \le 2, 1 \le j \le r_i\} \cup \{y_iy_{i+1} : 1 \le i \le m-3\}.$$

We define the labeling of $RW(m; n_{i,j}; r_i)$ by the function

$$f : V(G) \to \{1,2,,\sum_{i=1}^{2}\sum_{j=1}^{r_i} n_{ij} + 2(r_1 + r_2) + m\}$$

in two cases depending upon the order $m-2$ of the path y_i as follows:

Case 1: When m is odd.

$$f(b_{ij}) = (\sum_{k=1}^{r_1-j+1} n_{1,r_1-k+1})(2-i) + (\lceil\frac{m-2}{2}\rceil + \sum_{k=1}^{r_1} n_{1,k} + \sum_{k=1}^{j-1} n_{2,k} - 1)(i-1) + (-1)^i j + r_1 + 2,$$

for

$1 \le i \le 2, 1 \le j \le r_i + 1.$

$$f(c_{ij}) = \sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + \lceil\frac{m-2}{2}\rceil + 2r_1 + r_2 + (\lfloor\frac{m-2}{2}\rfloor - 1)(i-1) + (-1)^i j + 3,$$

for $1 \le i \le 2, 1 \le j \le r_i$.

$$f(x_{it}^l) = (\sum_{k=1}^{r_1} n_{1,k} + \sum_{k=1}^{i-1} n_{2,k} + \lceil\frac{m-2}{2}\rceil + r_1)(l-1) + (\sum_{k=1}^{r_1-i+1} n_{1,r_1-k+1} + r_1 + 2)(2-l) + (-1)^l(t+i),$$

for

$1 \le l \le 2, 1 \le i \le r_l, 1 \le t \le n_{li}$.

$$f(y_{2i-1}) = \sum_{k=1}^{r_1} n_{1,k} + r_1 + 1 + i, \text{ for } 1 \le i \le \lceil\frac{m-2}{2}\rceil.$$

$$f(y_{2i}) = \sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + \lceil\frac{m-2}{2}\rceil + 2r_1 + r_2 + i + 2,$$

for $1 \le i \le \lfloor\frac{m-2}{2}\rfloor$.

The edge weights under this labeling scheme forms a sequence of $\sum_{i=1}^{2}\sum_{j=1}^{r_i} n_{ij} + 2(r_1 + r_2) + m - 1$ number of consecutive integers

$$\{\sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + \lceil\frac{m-2}{2}\rceil + r_1 + r_2 + 4, \sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + \lceil\frac{m-2}{2}\rceil + r_1 + r_2 + 5, \ldots, 2\sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + 3(r_1 + r_2) + \lceil\frac{m-2}{2}\rceil + m + 2\}.$$

So, by Lemma 1, the labeling f admits *super edge magic total labeling*, where the magic constant in this case is

$$3\sum_{k=1}^{2}\sum_{s=1}^{r_k} n_{ks} + 5(r_1 + r_2) + \lceil\frac{m-2}{2}\rceil + 2m + 3.$$

Case 2: When m is even.

$$f(b_{ij}) = (\sum_{k=1}^{r_1} n_{1,k} + \sum_{k=1}^{j-1} n_{2,k} + r_1 + r_2 + m - 3)(i-1) + r_1 + 2 + (-1)^i j + (\sum_{k=1}^{r_1-j+1} n_{1,r_1-k+1})(2-i),$$

for

$1 \le i \le 2, 1 \le j \le r_i + 1.$

$$f(c_{ij}) = \sum_{k=1}^{r_1} n_{1,k} + (\frac{m-2}{2}) + (-1)^i j + (r_1 + r_2 + 1)(2-i) + (r_1 + 1),$$

for $1 \le i \le 2, 1 \le j \le r_i$.

$$f(x_{it}^l) = \sum_{k=1}^{r_1-i+1} n_{1,r_1-k+1}(2-l) + (\sum_{k=1}^{r_1} n_{1,k} + \sum_{k=1}^{i-1} n_{2,k} + r_1 + r_2 + m - 3)(l-1) + (-1)^l(i+t) + r_1 + 2,$$

for $1 \le l \le 2, 1 \le i \le r_l, 1 \le t \le n_{li}$.

$$f(y_{2i-1}) = \sum_{k=1}^{r_1} n_{1,k} + r_1 + i + 1, \text{ for } 1 \le i \le (\frac{m-2}{2}).$$

$$f(y_{2i}) = \sum_{k=1}^{r_1} n_{1,k} + (\frac{m-2}{2}) + 2r_1 + r_2 + i + 1,$$

for $1 \le i \le (\frac{m-2}{2})$.

The edge weights under this labeling scheme constitute a sequence of

$\sum_{i=1}^{2}\sum_{j=1}^{r_i} n_{ij} + 2(r_1 + r_2) + m - 1$ consecutive integers

$$\{\sum_{k=1}^{r_1} n_{1,k} + (\frac{m-2}{2}) + r_1 + r_2 + 3, \sum_{k=1}^{r_1} n_{1,k} + (\frac{m-2}{2}) + r_1 + r_2 + 3,$$

$$\ldots, 2\sum_{k=1}^{r_1} n_{1,k} + \sum_{k=1}^{r_2} n_{2,k} + 3(r_1+r_2) + (\frac{m-2}{2}) + m + 1\}.$$

Hence by Lemma 1, the labeling f can be extended to the *super edge-magic total labeling*, with magic constant in this case

$$3\sum_{k=1}^{r_1} n_{1,k} + 2\sum_{k=1}^{r_2} n_{2,k} + 5(r_1+r_2) + (\frac{m-2}{2}) + 2m + 2.$$

ow, we define a graph constructed by k copies of the *reflexive w-graph*, and we will prove in the next theorem that this graph is also super edge magic total.

Definition 3. The *reflexive w-tree* $RT(m;n_{i,j};r_i;k)$, for $1\le i\le 2k, 1\le j\le r_i$, is a graph obtained by joining k isomorphic copies of $RW(m;n_{i,j};r_i)$ with new vertices $a_i (1\le i\le k-1)$ by adding the edges $a_i b_{2i,r_{2i}+1}$ (or $a_i c_{2i,r_{2i}}$) and $a_i b_{2i+1,r_{2i}+1}$ for $1\le i\le k-1$.

Theorem 2. The graph $G\cong RT(m;n_{i,j};r_i;k)$, for $1\le i\le 2k, 1\le j\le r_i$ and $m,n_{i,j},r_i,k\in \mathbb{Z}^+$, is *super edge-magic total*.

Proof. The vertex and edge sets of $G\cong RT(m;n_{i,j};r_i;k)$ are defined as follows:

$V(G)=\{b_{i,j}:1\le i\le 2k, 1\le j\le r_i+1\}\cup\{c_{i,j}:$
$1\le i\le 2k, 1\le j\le r_i\}\cup$
$\{a_i:1\le i\le k-1\}\cup\{y_i^l:1\le i\le m-2, 1\le l\le k\}\cup$
$\{x_{i,t}^l:1\le l\le 2k, 1\le i\le r_l, 1\le t\le n_{li}\}.$

$E(G)=\{b_{i,j}c_{i,j}, b_{i,j+1}c_{i,j}:1\le i\le 2k, 1\le j\le r_i\}\cup$
$\{x_{i,t}^l c_{l,i}:1\le l\le 2k, 1\le i\le r_l, 1\le t\le n_{li}\}\cup$
$\{y_i^l y_{i+1}^l, y_1^l c_{2l-1,1}, y_{m-2}^l c_{21,1}:1\le i\le m-3, 1\le l\le k\}\cup$
$\{a_i b_{2i+1,r_{2i+1}+1}, a_i b_{2i,r_{2i}+1}, 1\le i\le k-1, \ \ for \ \ m\equiv 1(mod 2)\}\cup$

$\{a_i b_{2i+1,r_{2i+1}+1}, a_i c_{2i,r_{2i}}, 1\le i\le k-1, \ \ for \ \ m\equiv 0(mod 2)\}.$

The graph *G* is of order

$$\sum_{i=1}^{2k}\sum_{j=1}^{r_i} n_{i,j} + \sum_{i=1}^{2k}(2r_i+1) + k(m-1) - 1,$$

and size

$$\sum_{i=1}^{2k}\sum_{j=1}^{r_i} n_{i,j} + \sum_{i=1}^{2k}(2r_i+1) + k(m-1) - 2.$$

We define the labeling of $RT(m;n_{i,j};r_i;k)$

$$f:V(G)\to\{1,2,,\sum_{i=1}^{2k}\sum_{j=1}^{r_i} n_{ij} + \sum_{i=1}^{2k}(2r_i+1) + k(m-1) - 1\}$$

in two cases depending upon the order $m-2$ of the path y_i in the graph, as follows

Case 1: When m is odd.

$$f(b_{2i-1,j}) = \sum_{t=1}^{1-j+r_{2i-1}} n_{2i-1,1-t+r_{2i-1}} + \sum_{s=1}^{2i-2}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2i-2}(r_t+1)$$
$$+ r_{2i-1} + \lceil\frac{m-2}{2}\rceil(i-1) - j + 2,$$

for $1\le i\le k, 1\le j\le r_{2i-1}+1$.

$$f(b_{2i,j}) = \sum_{s=1}^{2i-1}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2i-1}(r_t+1) + \sum_{t=1}^{j-1} n_{2i,t} + i\lceil\frac{m-2}{2}\rceil + j,$$

for $1\le i\le k, 1\le j\le r_{2i}+1$.

$$f(c_{2i-1,j}) = \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2k}(r_t+1) + \sum_{t=1}^{2i-2} r_t + k\lceil\frac{m-2}{2}\rceil$$
$$+ (i-1)\lfloor\frac{m-2}{2}\rfloor + r_{2i-1} - j + i,$$

for $1\le i\le k$,
$1\le j\le r_{2i-1}$.

$$f(c_{2i,j}) = \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2k}(r_t+1) + \sum_{t=1}^{2i-1} r_t +$$
$$k\lceil\frac{m-2}{2}\rceil + i\lfloor\frac{m-2}{2}\rfloor + j + i - 1,$$

for $1\le i\le k, 1\le j\le r_{2i}$.

$$f(x_{it}^{2l-1}) = \sum_{s=1}^{2l-2}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2l-2}(r_t+1) + \sum_{t=1}^{1-i+r_{2l-1}} n_{2l-1,1-t+r_{2l-1}}$$
$$+ (l-1)\lceil\frac{m-2}{2}\rceil + r_{2l-1} - i - t + 1,$$

for $1\le l\le k, 1\le i\le r_{2l-1}, 1\le t\le n_{2l-1,i}$.

$$f(x_{it}^{2l}) = \sum_{s=1}^{2l-1}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2l-1}(r_t+1) + l\lceil\frac{m-2}{2}\rceil + \sum_{t=1}^{i-1} n_{2l,t} + i + t,$$

for $1\le l\le k, 1\le i\le r_{2l}, 1\le t\le n_{2l,i}$.

$$f(a_i) = \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + k\lceil\frac{m-2}{2}\rceil + \sum_{t=1}^{2k}(r_t+1) +$$
$$\sum_{t=1}^{2i} r_t + i\lfloor\frac{m-2}{2}\rfloor + i,$$

for $1\le i\le k-1$.

$$f(y_{2i-1}^l) = \sum_{s=1}^{2l-1}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2l-1}(r_t+1) + (l-1)\lceil\frac{m-2}{2}\rceil + i,$$

for $1 \le l \le k,\ 1 \le i \le \lceil \frac{m-2}{2} \rceil$.

$$f(y_{2i}^{l}) = \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2k}(r_t+1) + \sum_{t=1}^{2l-1} r_t + k\lceil \frac{m-2}{2} \rceil + (l-1)\lfloor \frac{m-2}{2} \rfloor + l + i - 1,$$

for $1 \le l \le k,\ 1 \le i \le \lfloor \frac{m-2}{2} \rfloor$.

The edge weights under this labeling scheme forms a sequence of $\sum_{i=1}^{2k}\sum_{j=1}^{r_i} n_{i,j} + \sum_{i=1}^{2k}(2r_i+1) + k(m-1) - 2$ number of consecutive integers

$$\{\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2k}(r_t+1) + k\lceil \frac{m-2}{2} \rceil + 2,\ \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2k}(r_t+1) + k\lceil \frac{m-2}{2} \rceil + 3,\ \ldots,\ , 2\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + 3\sum_{t=1}^{2k} r_t + k\lceil \frac{m-2}{2} \rceil + km + 3k - 1\}.$$

Hence by Lemma 1, the labeling scheme f can be extended to the *super edge-magic total labeling* of the graph $RT(m; n_{i,j}; r_i; k)$ with magic constant

$$3\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + 5\sum_{t=1}^{2k} r_t + k\lceil \frac{m-2}{2} \rceil + 2km + 4k - 1.$$

Case 2: When m is even.

$$f(b_{2i-1,j}) = \sum_{t=1}^{1-j+r_{2i-1}} n_{2i-1,1-t+r_{2i-1}} + \sum_{s=1}^{i-1}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{i-1}(r_{2t-1}+1) + \sum_{t=1}^{i-1} r_{2t} + (i-1)(\frac{m-2}{2}) + r_{2i-1} + 2 - j,$$

for $1 \le i \le k$, $1 \le j \le r_{2i-1}+1$.

$$f(b_{2i,j}) = \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s,t} + \sum_{s=1}^{i-1}\sum_{t=1}^{r_s} n_{2s,t} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{k} r_{2t} + \sum_{t=1}^{i} r_{2t-1} + \sum_{t=1}^{i-1}(r_{2t}+1) + \sum_{t=1}^{j-1} n_{2i,t} + i(\frac{m-2}{2}) + i + j - 1,$$

for $1 \le i \le k, 1 \le j \le r_{2i}+1$.

$$f(c_{2i-1,j}) = \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{s=1}^{i-1}\sum_{t=1}^{r_s} n_{2s,t} + \sum_{t=1}^{k} r_{2t} + \sum_{t=1}^{i} r_{2t-1} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{i-1}(r_{2t}+1) + (k+i-1)(\frac{m-2}{2}) + i - j.$$

for $1 \le i \le k, 1 \le j \le r_{2i-1}$,

$$f(c_{2i,j}) = \sum_{s=1}^{i}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{i}(r_{2t-1}+1) + (i-1)(\frac{m-2}{2}) + j,$$

for $1 \le i \le k, 1 \le j \le r_{2i}$.

$$f(x_{it}^{2l-1}) = \sum_{s=1}^{l-1}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{1-i+r_{2l-1}} n_{2l-1,1-k+r_{2l-1}} + \sum_{t=1}^{l-1}(r_{2t-1}+1) + \sum_{t=1}^{l-1} r_{2t} + (l-1)(\frac{m-2}{2}) + r_{2l-1} - i - t + 2,$$

for $1 \le l \le k, 1 \le i \le r_{2l-1}, 1 \le t \le n_{2l-1,i}$.

$$f(x_{it}^{2l}) = \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{k} r_{2t} + \sum_{t=1}^{l} r_{2t-1} + \sum_{t=1}^{i-1} n_{2l,t} + (i+k)\lceil \frac{m-2}{2} \rceil + i + t + l - 1,$$

for $1 \le l \le k, 1 \le i \le r_{2l-1}, 1 \le t \le n_{2l-1,i}$.

$$f(a_i) = \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{s=1}^{i}\sum_{t=1}^{r_s} n_{2s,t} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{i}(r_{2t}+1) + \sum_{t=1}^{k} r_{2t} + \sum_{t=1}^{i} r_{2t-1} + (i+k)(\frac{m-2}{2}) + i,$$

for $1 \le i \le k-1$.

$$f(y_{2i-1}^{l}) = \sum_{s=1}^{l}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{l}(r_{2t-1}+1) + \sum_{t=1}^{l-1} r_{2t} + (l-1)(\frac{m-2}{2}) + i,$$

for $1 \le l \le k, 1 \le i \le (\frac{m-2}{2})$.

$$f(y_{2i}^{l}) = \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{s=1}^{l-1}\sum_{t=1}^{r_s} n_{2s,t} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{l-1}(r_{2t}+1) + \sum_{t=1}^{k} r_{2t} + \sum_{t=1}^{l} r_{2t-1} + (k+l-1)(\frac{m-2}{2}) + i + l - 1,$$

for $1 \le l \le k, 1 \le i \le (\frac{m-2}{2})$.

The edge weights under this labeling scheme constitute a

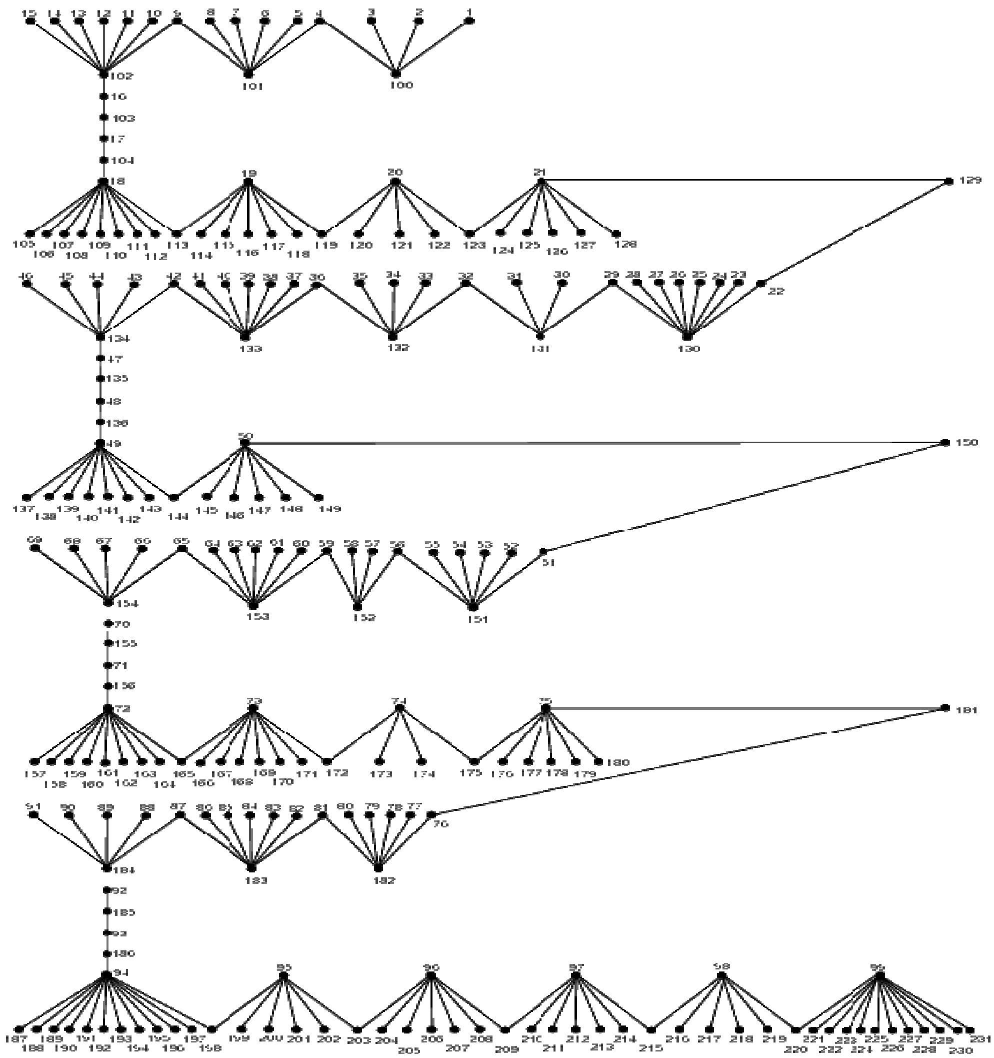

Figure 2: $RT(6;n_{i,j};r_i;4)$ for $1 \le i \le 8$, $1 \le j \le r_i$

sequence of $\sum_{i=1}^{2k}\sum_{j=1}^{r_i} n_{i,j} + \sum_{i=1}^{2k}(2r_i+1) + k(m-1) - 2$ number of consecutive integers

$$\{\sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{k} r_{2t} + k(\frac{m-2}{2}) + 2,\ \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{k} r_{2t} + k(\frac{m-2}{2}) + 3,$$

$$\ldots, \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{2k} r_t + \sum_{t=1}^{2k} r_t + \frac{km}{2} + km + k - 1\}$$

.Hence by Lemma 1, the graph $RT(m;n_{i,j};r_i;k)$ is *super edge magic total* with magic constant

$$2\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + 3\sum_{t=1}^{2k} r_t + \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \frac{km}{2} + 2km + 2k - 1.$$

Now, we give a corollary which explains the relationship between the reflexive w-graphs and reflexive w-trees.

Corollary 1. For $k = 0$, the reflexive w-tree $RT(m;n_{i,j};r_i;0)$, is the same as the reflexive w-graph $RW(m;n_{i,j};r_i)$, for $1 \le i \le 2$, $1 \le j \le r_i$.

Example 1. The super edge magic total labeling of the reflexive w-tree $RT(6;n_{i,j};r_i;4)$ for $1 \le i \le 8$, $1 \le j \le r_i$, is presented in the figure 2. Here $m = 6$ and $k = 4$.

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 2 \\ 4 \\ 4 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} n_{11}, n_{12}, n_{13} \\ n_{21}, n_{22}, n_{23}, n_{24} \\ n_{31}, n_{32}, n_{33}, n_{34}, n_{35} \\ n_{41}, n_{42} \\ n_{51}, n_{52}, n_{53}, n_{54} \\ n_{61}, n_{62}, n_{63}, n_{64} \\ n_{71}, n_{72}, n_{73} \\ n_{81}, n_{82}, n_{83}, n_{84}, n_{85}, n_{86} \end{pmatrix} = \begin{pmatrix} 5,4,2 \\ 7,5,3,4 \\ 3,5,3,2,6 \\ 6,4 \\ 3,5,2,4 \\ 7,6,2,4 \\ 3,5,4 \\ 10,4,5,5,4,10 \end{pmatrix}$$

The edge weights form the sequence {101,102,103,,330} and hence by Lemma 1, this labeling can be extended to the super edge magic total labeling with magic constant 562.

3 CONCLUDING REMARKS

In this paper, we constructed new class of trees referred as *reflexive w-trees*. These graphs are constructed using *w-trees* introduced by Javaid et al. [10]. We studied the *SEMT labeling* of these graphs. It is an intresting and challenging problem to study the SEMT labeling of trees due to the conjecture by Enomoto et al. [3], which states that all trees are *super edge magic total*. We invite the readers to investigate

- The SEMT labeling of all trees and give a general proof.
- The SEMT labeling of disjoint union of reflexive w-trees.
- The SEMT labeling of disjoint union of stars and reflexive w-trees.
- The SEMT labeling of one vertex amalgamation of k copies of reflexive w-trees.

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