HE TRICYCLIC GRAPHS WITH THE EXTREMUM ZEROTH-ORDER GENERAL RANDIC INDEX

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ABSTRACT: A \((n, n+2)\)-graph \(G\) is a connected simple graph with \(n\) vertices and \(n + 2\) edges. If \(d_v\) denotes the degree of the vertex \(v\), then the zeroth-order general Randic index of the graph \(G\) is defined as \(R^0(G) = \sum_{v \in V} d_v^\alpha\), where \(\alpha\) is a real number. We characterize, for any \(\alpha\), the \((n, n+2)\)-graphs with the extremum (minimum for \(\alpha < 1\) and maximum for \(\alpha < 0\) or \(\alpha > 1\)) zeroth-order general Randic index of \((n, n + 2)\)-graphs.

Keywords: \((n, n+2)\)-graph, zeroth-order general Randic index, degree sequence

1 INTRODUCTION

Let \(G = (V, E)\) be a simple connected graph with the vertex set \(V\) and edge set \(E\). For any \(v \in V\), \(N(v)\) denotes the neighborhood of \(v\) and \(d_v = |N(v)|\) is the degree of \(v\). The Randic index was defined [1] as, and then generalized by replacing the exponent \(-\frac{1}{2}\) by an arbitrary real number \(\alpha\) [2]. This graph invariant is called the general Randic index and denoted by \(R_\alpha\), i.e. \(R_\alpha(G) = \sum_{u,v \in E} (d_u d_v)^\alpha\). Some results of the general Randic index can be found in [3].

Kier and Hall [4] defined the zeroth-order Randic index as \(R^0(G) = \sum_{v \in V} d_v^{-1/2}\). Eventually LI and Zheng [5] defined the zeroth-order general Randic index of graph \(G\) as \(R^0_\alpha(G) = \sum_{v \in V} d_v^{-\alpha}\), for any real number \(\alpha\). Chen and Deng [6] characterized the \((n, n+1)\)-graph with the maximum and minimum zeroth-order general Randic index for any real number \(\alpha\).

Let \(G(k,n)\) be the set of connected simple graphs have \(n\) vertices and the minimum degree of vertices is \(k\). Pavlovic and Gutman [7] used linear programming formulation and calculated the minimum value of Randic index for \(G(1,n)\). For \(k = 2\) the minimum value of Randic index was obtained in [8]. Li et al. [9] studied the famous conjecture of minimum Randic index for \(G(k,n)\), however, faced with a serious problem [10]. In this paper, we investigate the zeroth-order general Randic index \(R^0_\alpha(G)\) of \((n, n+2)\)-graphs \(G\), i.e., connected simple graphs with \(n\) vertices and \(n+2\) edges.
We use the classification of \((n, n+2)\)-graphs given in [11] and find the extremum of \(R_0^0(G)\) in all disjoint classes.

Then, the extremum of \(R_0^0(G)\) of \((n, n + 2)\)-graphs can be found by comparing the extremum of \(R_0^0(G)\) of all classes. Note that if \(\alpha = 0\) then \(R_0^0(G) = n\), and if \(\alpha = 1\) then \(R_0^0(G) = 2m\). Therefore in the following we always assume that \(\alpha \neq 0, 1\).

### 2 Increasing and decreasing transformations

For convenience, we use some transformations given in [6] and some new results in order to increase or decrease the zeroth-order general Randić index.

#### Transformation 1

If there are two vertices \(u \) and \(v \) in \(G\) such that \(d_u = p \geq 1\), \(d_v = q \geq 1\) and \(p \leq q\), in which, the vertices \(u_1, u_2, \ldots, u_k\) are adjacent to \(u\), then we put \(G' = G - \{u u_1, u u_2, \ldots, u u_k\} + \{v u_1, v u_2, \ldots, v u_k\}\), \(1 \leq k \leq p\), as shown in Figure 1.

**Lemma 2.1.** For the two graphs \(G\) and \(G'\) defined in Transformation 1, we have

1. \(R_0^0(G') > R_0^0(G)\) for \(\alpha < 0\) or \(\alpha > 1\).
2. \(R_0^0(G') < R_0^0(G)\) for \(0 < \alpha < 1\).

**Proof:** See Ref [6].

#### Transformation 2

Let \(uv\) be an edge \(G\), the degree \(d_G(u)\) of \(u\) is \(p \geq 1\), \(N_G(v)\) is the neighborhood of \(v\) and \(N_G(v) - u = \{w_1, w_2, \ldots, w_k\}\). We put

\[ G' = G - \{v w_1, v w_2, \ldots, v w_k\} + \{u w_1, u w_2, \ldots, u w_k\}, \]

as shown in Figure 2.

**Lemma 2.2.** For the two graphs \(G\) and \(G'\) were defined in Transformation 2, we have

1. \(R_0^0(G') > R_0^0(G)\) for \(\alpha < 0\) or \(\alpha > 1\).
2. \(R_0^0(G') < R_0^0(G)\) for \(0 < \alpha < 1\).

**Proof:** By the definition of \(R_0^0(G)\), we have

\[ \Delta = R_0^0(G') - R_0^0(G) = [\{p + 1\}^\alpha + 1^\alpha] - [p^\alpha + 2^\alpha] = [\{(p + 1)^\alpha - p^\alpha\} - [2^\alpha - 1^\alpha]] = \alpha(\xi - \eta^{a-1}) \]

where \(\eta \in (1, 2), \xi \in (p, p + 1)\) and \(d_G(x_i) = p\).

Since \(p \geq 2\) therefore \(\xi > \eta\) and therefore \(\Delta < 0\) for \(0 < \alpha < 1\) and \(\Delta > 0\), for \(\alpha > 0\) or \(\alpha > 1\).

#### Transformation 3

Let \(G\) be a graph include subgraph \(G_0\) and the interior path \(W: x_1x_2x_3\), with \(d_G(x_2) = 2\), and vertices \(x_1\) and \(x_3\) do not joined in \(G_0\) as shown in Figure 3.

**Lemma 2.3.** For the two graphs \(G\) and \(G'\) were defined in Transformation 3, we have

1. \(R_0^0(G') > R_0^0(G)\) for \(\alpha < 0\) or \(\alpha > 1\).
2. \(R_0^0(G') < R_0^0(G)\) for \(0 < \alpha < 1\).

**Proof:** See Ref [6].

**Figure 3: Transformation 3**

**Figure 4: Lemma 2.4**
Proof: Note that
\[ \Delta = R^0_a(G) - R^0_a(G') = [(q + k + 1)^a + 2^a] - [(q + 1)^a + (k + 2)^a] = (q + k + 1)^a - (q + 1)^a -\]
\[\alpha k (\xi^{-a-1} - \eta^{-a-1})\]
Where \(d_G(u) = q, \ \eta \in (2, k + 2), \ \xi \in (q + 1, q + k + 1).\) If \(q + 1 > k + 2.\) Therefore \(\Delta < 0\) for \(0 < \alpha < 1\) and \(\Delta > 0\) for \(\alpha > 1.\) On the other hand
\[\Delta = R^0_a(G) - R^0_a(G') = [(q + k + 1)^a + 2^a] -
\[(q + 1)^a + (k + 2)^a] = [(q + k + 1)^a - (k + 2)^a]\]
Where
\[-(q + 1)^a - 2^a = \alpha(q - 1)(\xi^{-a-1} - \eta^{-a-1})\]
\(d_G(u) = q, \ \eta \in (2, q + 1), \ \xi \in (k + 2, q + k + 1).\) If \(k + 2 > q + 1.\)
Therefore \(\Delta < 0\) for \(0 < \alpha < 1\) and \(\Delta > 0\) for \(\alpha < 0,\) \(\alpha > 1.\)

3 The extremum zeroth-order general Randic index of \((n, n + 2)\)-graphs

Let \(G(n, n + 2)\) be the set of simple connected graphs with \(n\) vertices and \(n + 2\) edges. In this chapter, we use the classification of \(G(n, n + 2)\) given in [11] and find an extremum zeroth-order general Randic index \(R^0_a(G)\) of each class of \(G(n, n + 2)\) (see Figure 5). The extremum (minimum for \(0 < \alpha < 1\) and maximum for \(\alpha < 0, \alpha > 1\)) of \(R^0_a(G)\) will obtain by comparing the extremum of \(R^0_a(G)\) of all classes.

To this end, we use three increasing/decreasing transformations 1, 2, 3 and the lemma 2.4, have been described in previous chapter. By using and repeating these transformations we will increase/decrease \(R^0_a(G)\) of each class of \(G(n, n + 2)\) as much as possible.

Initially, repeating transformation 2, any graph \(G\) in \(G(n, n + 2)\) can be changed into a graph, in which, the edges not on the cycles are pendant edges. In the second step, by using Transformation 1, we reach a graph, in which, the pendant edges have been attached to a single vertex. Then we apply the Transformation 3 to minimize the length of cycles. Transformation 3 is repeated till it is possible. This step reduces the cycles to \(C_3\) or \(C_4.\) Now, we have a graph that all its pendant edges were attached to a set of vertices. Once again, applying the Transformation 1, we reach a graph, in which, all pendant edges have been attached to a single vertex. By using the Lemma 2.4, we give up some cases (those their pendant edges have been attached to the vertex of degree 2 of cycles). Now we calculate the \(R^0_a(G)\) of extremal graphs in each class.

All nineteen classes of \(G(n, n + 2)\) have been Shawn in the second column of Figure 5, the third column represents the final graph(s) obtained by using the above process in each classes as described.

<table>
<thead>
<tr>
<th>Class</th>
<th>Original graph</th>
<th>Final graph(s)</th>
<th>Extremum/General Randic index</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>((n - 2) + 6 \times 2^a + (n - 7))</td>
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<tr>
<td>2</td>
<td></td>
<td></td>
<td>((n - 2) + 5 \times 2^a + 4^a + (n - 7))</td>
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<td>3</td>
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<td>((n - 0) + 5 \times 2^a + 3^a + (n - 8))</td>
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<td>4</td>
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<td>((n - 0) + 4 \times 2^a + 3^a + (n - 8))</td>
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<td>((n - 0) + 2 \times 2^a + 3^a + (n - 8))</td>
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</tbody>
</table>

Figure 5: Original graph, Final graph(s) and The extremum zeroth-order general Randic index in 19 classes.

The fourth column represents the extremum zeroth-order general Randic index of classes (maximum for \(0 < \alpha < 1\) and minimum for \(\alpha < 0\) or \(\alpha > 1\)). For any real number \(\alpha\), the extremum zeroth-order general Randic index can be found by comparing the values of forth column. As an example, the maximum zeroth-order general Randic index for \(\alpha = 0.5\) and \(n = 15\) is equals 21,509,861 which is related to class 7.
REFERENCES