A HAM SOLUTION OF NON-LINEAR CONVECTION-RADIATIVE HEAT TRANSFER EQUATION

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ABSTRACT: *The aim in this paper is to approach the approximation solution of transfer of heat problem and to develop the results via HAM which is applied to a non-Fourier, convective-radiative equation of heat transfer having coefficients of specific heat as variable. The HAM results are then comparable with those provided by the established technique, the HPM and Runge-Kutta method 4th order in order to confirm the efficiency of the utilized method. The applied technique showed its effectiveness and reliability asof compared to the other traditional perturbation method.*

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1. INTRODUCTION

The phenomena existing in this world are fundamentally non-linear and delivered by non-linear equations. Since the existence of the performance-based digital computers enabled it to deal with a linear problem. But still, strictly speaking, it proved to be difficult to extract an accurate solution of non-linear phenomenological problem. Particularly it is usually much hard to find an analytical approximate solution than finding a numerical one for a nonlinear equation, however, we now make use of highly performance showing super computers and some superior quality computational symbolic programming softwares like Maple, Mathematica, and so on.
The usefulness of the nonlinear pdes cannot be denied in describing the different phenomenae in various scientific disciplines. Despite of a few numbers of such problems, some of them do not possess a quite perfect analytical solution. Hence such nonlinear problems must be dealt with the approximate methods. In 1992, S. Liao developed the comprehensive idea of the HAM [1-2] and then modified it from time to time [3].
The success of this method is that it is successfully applicable to solve many kinds of nonlinear equations with a great convergence rate when it is compared with other technique [4]. This method doesn't depend upon any small or large parameter. Furhter it is mostly valid for models possessing nonlinearity [5]. HAM is quiet dissimilar to all preceding numerical techniques since it avails the benefit of an auxiliary parameter h, and this h offers us a trouble-free approach for adjustment and controling the region of convergence for series solution.
The convective heat transfer, often referred to simply as convection, is by virture of the movement of fluids for the transfer of heat from one place to another. Convection is usually the dominant from of heat transfer in liquids and gases. Although often discussed as a distinct method of transfer of heat, the convective heat transfer involves the combined processes of conduction and advection.
The convection as a terminology, sometimes may refer to heat transfer by any fluid motion, however, the advection is more precisely used term for the transfer by virtue of nly bulk flow of fluid. The transfer of heat process from a fluid to the fluid, or the reverse, is not only because of these transfers, but also conduction and diffusion of heat through the boundary layer near to the solid. Therefore, a process requires heat advection and diffusion as well, without a moving fluid, such a process that is termed as convection.
The heat transfer principles in technological systems may be applicable to the body of a human to find our how the heat is transferred by the body. Because of the continuous metabolism of nutrients, the heat is produced in the body for the systems of the body, the energy is provided by this process. To keep the body from overheating, the excess of heat shouldbe dissipated from it. The human body requires extra fuel which increases the rate of heat production and the metabolic rate if a man is engaged in elevated level of physical motion. The additional heat produced must be removed from the body by use additional methods so as to keep a healthy level internal temperature.
In [6], the model of a convective-radiative non-Fourier conduction transfer of heat problem having variable coefficient of heat has been studied by applying homotopy perturbation method which was proved to be more effective and convenient as compared with the results produced with the 4th order Runge-Kutta method for the verification of the accuracy of the used method. The HAM is also established its reliability and efficiency for dealing with the temperature distribution in system of combined radiation-convection.

2. THE PROBLEM FORMULATION

Let us consider the model of combined radiative- convective cooling of system having a low temperature. Suppose the system posseses a volume V, areaof surface A_s, specific heat r, the density τ, and initial temperature T_i, at $t=0$, A convective environment at the temperature T_a, the system is exposed with a convective transfer of heat coefficient c. Through radiation, the system is supposed to lose heat and the sink temperature is defined to be T_s. Let us suppose furhter that the specific heat r is a linear temperature function in the following form:

$$r = r_a\left[1+\alpha\left(T-T_a\right)\right], \quad (1)$$

Where r_a is termed as the specific heat at T_a and α is considered as constant. Making use of the non-Fourier heat conduction, the the initial condition and the cooling equation are:

$$\tau Vr\frac{dT}{dt}+\tau Vrk\frac{d^2T}{dt^2}+cA_s\left(T-T_a\right)+E\Omega A_s\left(T^4-T_s^{\,4}\right)=0, \quad (2)$$

$$t=0, \quad T=T_i, \quad (3)$$

$$t=0, \quad \frac{dT}{dt}=0. \quad (4)$$

For the solution of equation (1) we can change the following parameters:

$$Q=\frac{T}{T_i},\ Q_a=\frac{T_a}{T_i},\ Q_s=\frac{T_s}{T_i},\ x=\frac{t}{\frac{\tau Vr_a}{cA_s}},\ \varepsilon_1=\alpha T_i,$$

$$\varepsilon_2=\frac{E\Omega T_i^3}{c},\ \Psi=\sqrt{\frac{k}{\frac{\tau Vr_a}{cA_s}}}. \quad (5)$$

After changing the parameter, the heat transfer equation will be:

$$\left[1+\varepsilon_1\left(Q-Q_a\right)\right]\frac{dQ}{dx}+\left[1+\varepsilon_1\left(Q-Q_a\right)\right]\Psi^2\frac{d^2Q}{dx^2} \quad (6)$$
$$+\left(Q-Q_a\right)+\varepsilon_2\left(Q^4-Q_s^{\,4}\right)=0,$$

$$x=0, \quad Q=1, \quad (7)$$

$$x=0, \quad \frac{dQ}{dx}=0. \quad (8)$$

For simplification, we consider the form of $Q_a=Q_s=0.$ so we have:

$$\left[1+\varepsilon_1 Q\right]\frac{dQ}{dx}+\left[1+\varepsilon_1 Q\right]\Psi^2\frac{d^2Q}{dx^2}+Q+\varepsilon_2Q^4=0, \quad (9)$$

$$x=0, \quad Q=1, \quad (10)$$

$$x=0, \quad \frac{dQ}{dx}=0. \quad (11)$$

We define a non linear term as

$$N[Q(x;p)]=(1+\varepsilon_1Q(x;p))\frac{dQ}{dx}(x;p)+Q(x;p)$$
$$+(1+\varepsilon_1Q(x;p))\Psi^2\frac{d^2Q}{dx^2}(x;p)+\varepsilon_2Q^4(x;p) \quad (12)$$

Then the approximate solution is defined to be as;

$$Q(x)=\sum_{n=0}^{\infty}Q_n(x) \quad (13)$$

Thus as stated by *Homotopy Analysis Method (HAM)*, the zero order deformation equation,

$$(1-p)L\left[Q(x;p)-Q_0(x)\right]=ph\mathrm{H}(x)\mathrm{N}\left[Q(x;p)\right] \quad (15)$$

By using $p=0$ and $p=1$ in equation (15) respectively, then we have $Q(x;0)=Q_0(x)$ and the equation become the initial guess for the given problem, whereas $Q(x;1)=Q(x)$ is the exact solution. Then the solution of the given equation takes the following form,

$$Q(x)=Q_0(x)+\sum_{n=1}^{\infty}p^nQ_n(x), \quad (16)$$

where $Q_n(x)=\frac{1}{n!}\frac{\partial^nQ(x)}{\partial p^n}$, Is calculating at $p=0$, exist for $n\geq 1$, and also converges at $p=1$. Then the solution of equations under study become,

$$Q(x)=Q_0(x)+Q_1(x)+Q_2(x)+..., \quad (17)$$

Now according to the Homotopy Analysis Method (HAM), the higher order deformation equation for the given system of equations becomes as,

$$L\left[Q_n(x)\right]=\chi_nL\left[Q_{n-1}\right]=hH(x)R_n(Q_{n-1}), \quad (18)$$

Applying the inverse operator, we may extract the nth order deformation equation solution as follows,

$$Q_n=\chi_nQ_{n-1}+h\int_0^x H.R_n\left(Q_{n-1}\right)dt, \quad (19)$$

Now for $n=1,2,...,$ we get the following sets of equations:

$$Q_1(x)=h\int_0^x H.R_1\left(Q_0\right)dx. \quad (20)$$

$$Q_2(x)=Q_1+h\int_0^x H.R_2\left(Q_1\right)dx. \quad (21)$$

$$Q_3(x)=Q_2+h\int_0^x H.R_3\left(Q_2\right)dx. \quad (22)$$

And so on.

Since
$$R_n\left(Q_{n-1}\right)=\left.\frac{\partial^{n-1}\left[N\left[Q(x,t;p)\right]\right]}{(n-1)!\partial p^{n-1}}\right|_{p=0} \quad (23)$$

Now putting $n=1$, in (23), we have the following forms of the above quantities,

$$R_1\left(Q_0\right)=Q_{0x}+\varepsilon_1Q_0Q_{0x}+\Psi^2Q_{0xx}+\varepsilon_1\Psi^2Q_0Q_{0xx}+Q_0+\varepsilon_2Q_0^{\,4} \quad (24)$$

Now putting $n=2$, in (23), we get

$$R_2(Q_1)=Q_{1x}+\varepsilon_1Q_0Q_{1x}+\varepsilon_1Q_1Q_{0x}+\Psi^2Q_{1xx}$$
$$+\varepsilon_1\Psi^2Q_0Q_{1xx}+\varepsilon_1\Psi^2Q_1Q_{0xx}+Q_1+4\varepsilon_2Q_0^{\,3}Q_1 \quad (25)$$

Using all these calculated values in (20), and (21) respectively, we get,

$$Q_1(x)=h\int_0^x H(x)\Big(Q_{0x}+\varepsilon_1Q_0Q_{0x}+\Psi^2Q_{0xx}+$$
$$\varepsilon_1\Psi^2Q_0Q_{0xx}+Q_0+\varepsilon_2Q_0^{\,4}\Big)dx, \quad (26)$$

$$Q_2(x)=Q_1+h\int_0^x H(x)\Big(Q_{1x}+\varepsilon_1Q_0Q_{1x}+\varepsilon_1Q_1Q_{0x}$$
$$+\Psi^2Q_{1xx}+\varepsilon_1\Psi^2Q_0Q_{1xx}+\varepsilon_1\Psi^2Q_1Q_{0xx} \quad (27)$$
$$+Q_1+4\varepsilon_2Q_0^{\,3}Q_1\Big)$$

And so on.

For the purpose of the solution of (13), we consider the two possible cases here

Case 1: If $-1+4\Psi^2>0$, then we start with the initial guess as,

$$Qo:=\frac{1}{2}\frac{\left(1+\sqrt{-4\Psi^2+1}\right)e^{\frac{1}{2}\frac{\left(-1+\sqrt{-4\Psi^2+1}\right)x}{\Psi^2}}}{\sqrt{-4\Psi^2+1}} \tag{28}$$

$$+\frac{1}{2}\frac{\left(-1+\sqrt{-4\Psi^2+1}\right)e^{\frac{1}{2}\frac{\left(-1-\sqrt{-4\Psi^2+1}\right)x}{\Psi^2}}}{\sqrt{-4\Psi^2+1}}$$

Case 2: If $-1+4\Psi^2<0$, we get:

$$Q_0=\frac{e^{-\frac{1}{2}\frac{x}{\Psi^2}}\sin\left(\frac{1}{2}\frac{\sqrt{-4\Psi^2+1}x}{\Psi^2}\right)}{\sqrt{-4\Psi^2+1}} \tag{29}$$

$$+e^{-\frac{1}{2}\frac{x}{\Psi^2}}\cos\left(\frac{1}{2}\frac{\sqrt{-4\Psi^2+1}x}{\Psi^2}\right)$$

3. THE RESULTS AND DISCUSSIONS

In this section, we discuss the results provided by HAM to demonstrate the effectiveness of the method. We take various numerical values of $\varepsilon_1, \varepsilon_2$, and x, and then a comparison is made with the results givn by the HPM and 4th order Runge-Kutta technique.The numerical results showed that this method has very good accuracy as compared with the HPM.

For the values of the variable $\varepsilon_1, \varepsilon_2$, the results of the our work are analysed and presented in tabulated form with the munarical results found by HPM and 4^{th} order Runge-Kutta in table 1 and table 2.

A quiet interesting accordance among the results is observable, confriming the outstanding strength of the HAM applicability. As mentioned in table 1, the difference between the results of HAM and numerical one is more remarkable. For large ε_1, and with small ε_2, this difference is significantly becomes large, and as ε_2, roughly reaches ε_1, the error of the found results are reduced then.

In the case of Ψ, variable, the findings of the present techniques are given tabular forms with the numerical results produced from 4th-order Runge-Kutta technique in table 3. The verification of the accuracy is easy from the results thus obtained. The graphical results are also shown here.

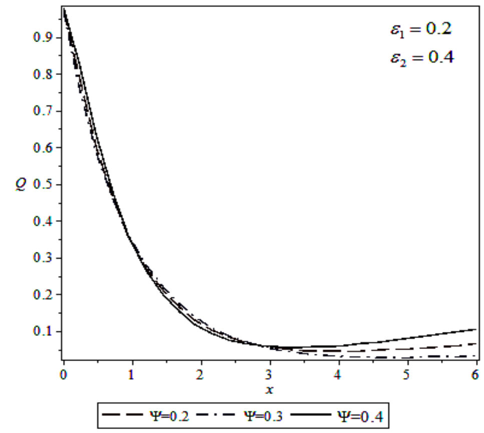

Figure 1: Result of HAM for first case for different values of Ψ and $\varepsilon_1 = 0.2, 0.4$.

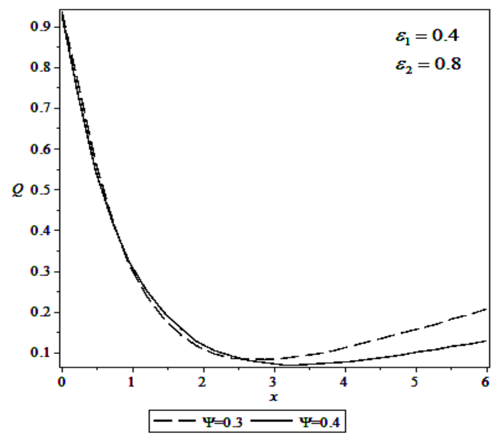

Figure 2: Result of HAM for first case for different values of Ψ and $\varepsilon_1 = 0.4, 0.8$.

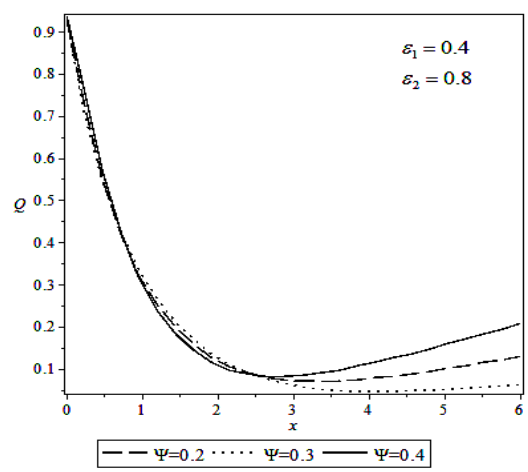

Figure 3: Result of HAM for second case for different values of Ψ and $\varepsilon_1 = 0.4, 0.8$.

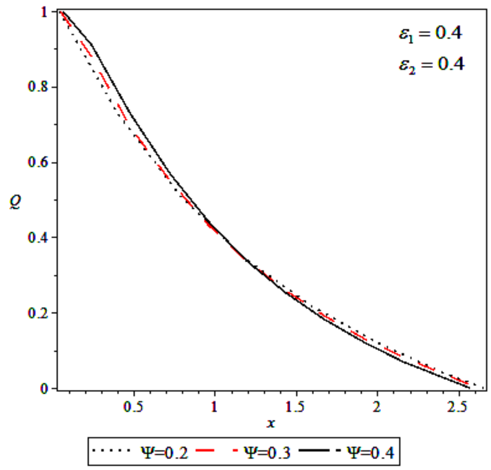

Figure 4: Result of HAM for second case for different values of Ψ and $\varepsilon_1 = 0.4, 0.4$.

Table 1: The result of HAM compare with HPM and fourth order Runge – Kutta method at $x = 1$ and $\Psi = 0.2$

ε_1	ε_2	HAM	RK4	HPM-RK	HAM-RK
0.4	0.4	0.4067523511	0.4074278881	0.0009364344	0.0006755370
0.4	0.6	0.3866760736	0.3863742390	0.0010513435	0.0003018346
0.4	0.8	0.3686118220	0.3684492318	0.0069269788	0.0001625902
0.8	0.4	0.4815175140	0.4804393217	0.0023393815	0.0010781923
0.8	0.6	0.4551749004	0.4561975967	0.0065121102	0.0010226963
0.8	0.8	0.4342652312	0.4353383524	0.0062834029	0.0010731212

Table 2: The result of HAM compare with HPM and fourth order Runge – Kutta methodat $x = 1$ and $\Psi = 0.8$

ε_1	ε_2	HAM	RK4	HPM-RK	HAM-RK
0.4	0.4	0.5766843599	0.5777179513	0.0019204415	0.0010335914
0.4	0.6	0.5429731890	0.5379133117	0.0063213087	0.0050598773
0.4	0.8	0.5110667186	0.5010762419	0.0188232983	0.0099904767
0.8	0.4	0.6783670530	0.6568938197	0.0415231859	0.0214732333
0.8	0.6	0.6105870183	0.6214500824	0.0294167365	0.0108630641
0.8	0.8	0.5854216838	0.5882388366	0.0123149693	0.0028171528

Table 3: The result of HAM compare with HPM and fourth order Runge – Kutta method at $x = 1, \varepsilon_1 = 0.4,$ and $\varepsilon_2 = 0.4$

Ψ	HAM	RK4	HPM-RK	HAM-RK
0.1	0.4066517963	0.4073399330	0.0007848148	0.0006881367
0.3	0.4086785430	0.4089328577	0.0013167486	0.0002543147
0.7	0.5281471839	0.5266888944	0.0022185847	0.0014582895
0.9	0.6266955173	0.6262974076	0.0016002882	0.0003981097

CONCLUSION:

The homotopy analysis method has been very much connected on the non-linear On Non-Linear Convective-Radiative Heat Transfer Equation for discovering the inexact arrangements. The relationship made between exact solution and the HAM demonstrates that HAM is almost near the careful arrangement, and it is exceptionally powerful and exact as exhibited by Table 1,2,3. Further on the off chance that we take $h = -1$, we can get the after effects of HPM as an extraordinary instance of the HAM.

The HAM has the non-zero auxiliary parameter h , by method for which we can control and change the union zone of the series solutions. Not at all like the other numerical methods, has it given a decent level of exactness for taking care of high nonlinear issues. Clearly, it is inferred that the HAM is an extremely dependable, effective and capable device with the assistance of which we can tackle profoundly non- linear problems in science and engineering with no impediments and suspicions.

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