MULTI-LEVEL DISTANCE LABELINGS FOR GENERALIZED PETERSEN GRAPH $P(n,3)$

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ABSTRACT: Let G be a connected graph with diameter diam(G) and $d(x,y)$ denotes the shortest distance between any two distinct vertices x,y in G. Radio labeling (multi-level distance labeling or distance labeling) of G is a one-to-one mapping $f:V(G)\to Z^{+}\cup\{0\}$ satisfying $d(x,y)+|f(x)-f(y)|\geq diam(G)+1$ for all $x,y\in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of a graph G. The radio number of G denoted by rn(G), is the lowest span over all radio labelings of the graph. In this paper, we establish the radio number for the generalized Petersen graph $P(n,3)$ when $n=6k+4$.

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1. INTRODUCTION

A radio labeling (multi-level distance labeling) is an assignment of label, denoted by integers, to the vertices of graph. Let $G=(V(G),E(G))$ be a connected graph with a vertex set $V(G)$ and an edge set $E(G)$. A radio labeling is a function from the vertices of the graph to some subset of non-negative integers. The task of radio labeling is to assign to each station a non-negative smallest integer such that the interference in the nearest channel should be minimized. In 1980 [3], Hale presented this channel assignment for the very first time by relating it to the theory of graphs. Later in 2001, Chartrand et al. [1] applied this idea for assignment of channels to FM radio station. Liu and Zhu completely studied the radio numbers for Paths and cycles in [8].

The radio number for the square (adding edges between vertices of distance two apart) of paths was completely determined by Liu and Xie [7] who also discussed the problem for the square of a cycle [6]. M. T. Rahim and I. Tomescu in [9], discussed a helm graph which is obtained from a wheel by attaching a vertex of degree one to each vertex of the cycle of the wheel and determined its radio number for every $n\geq 3$, $\mathrm{rn}(H_3)=13$, $\mathrm{rn}(H_4)=21$ and $\mathrm{rn}(H_n)=4n+2$ for any $n\geq 5$. They also proposed a lower bound of rn(G) related to the length of a maximum Hamiltonian path in the graph of distance of G. Lower bound for the generalized gear graph $J_{t,n}$, which is obtained from a wheel graph by introducing t vertices between every pair of adjacent vertices on the cycle was found by M. T. Rahim et al. in [10]. Radio number for different families of graphs have been investigated in [11], [12], [13], [14], [15] and the reference therein.

For a simple graph G, distance between any distinct pair of vertices in G denoted by $d(x,y)$ is the length of the smallest path between them. The diameter of G, diam$(G)=d$, is the maximum shortest distance between any two distinct vertices in G.

A radio labeling is a one-to-one mapping $f:V(G)\to Z^{+}\cup\{0\}$ satisfying the condition

$$|f(x)-f(y)|\geq \mathrm{diam}(G)+1-d(x,y)$$

for any pair of vertices x, y in G.

The largest number that f maps to a vertex of a graph is the span of labeling f. Radio number of G is the minimum span taken over all radio labelings of G and is denoted by rn(G).

In this paper, radio number of the graphs $\mathrm{P}(n,3)$, $n=6k+4$ are determined. The main theorems of this paper is:

Theorem 1.
For the generalized Petersen graphs, $P(n,3)$, $n=6k+4$ and $k=2s+1$ where $s\geq 2$.

$$rn(P(n,3))=\frac{6k^2+33k+19}{2}.$$

Theorem 2.
For the generalized Petersen graphs, $\mathrm{P}(n,3)$, $n=6k+4$ and $k=2s$ where $s\geq 3$

$$rn(P(n,3))=\frac{6k^2+39k+22}{2}.$$

2. *PRELIMINARIES*

A generalized Petersen graph, $P(n,m)$; $n \geq 3$ and $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ has a vertex set

$V(G)=\{u_i, v_i : i=1,2,...n\}$

and an edge set $E(G)=\{u_iu_{i+1}, v_iv_{i+m}, u_iv_i \mid$ with indices taken modulo $n\}$

Remark 1.[4]

The diameter of $P(n,3)$ is

diam $(P(n,3)) = d = k+4, if\ n = 6k+4$

3. LOWER BOUND FOR $P(n,3)$

In this section, the lower bound for $\mathrm{rn}\left(\mathrm{P}\left(\mathrm{n},3\right)\right)$, where $n=6k+4$ are determined. For this purpose, first examine the maximum possible sum of the pairwise distance between any three vertices of $P(n,3)$ and use this maximum sum to develop a minimum possible gap between the i^{th} and $(i+2)^{nd}$ a largest label. Using 0 for the smallest label and taking the size of gap into account then provides a lower bound for the span of any labeling.

Lemma 3.

Let $P(n,3)$ be the family of generalized Petersen graphs, $\mathrm{n}=6\mathrm{k}+4.$

(i) For each vertex u_1 on the outer cycle there is exactly one vertex at a distance d , diameter of $P(n,3)$.

(ii) For each vertex v_1 on the inner cycle there is exactly one vertex at a distance d , diameter of $P(n,3)$.

Proof.

(i) We show that $d(u_1,u_{3k+3}) = k+4 = d$. Since $n=6k+4$, there are equal number of vertices on the left half and right half of the cycle.

The path from u_1 to u_{3k+3} is of length $k+4$ as

$$u_1 \to v_{3(0)+1} \to v_{3(1)+1} \to v_{2(2)+1} \to ...$$
$$v_{3(k)+1} \to u_{3k+1} \to u_{3k+2} \to u_{3k+3}$$

(ii) $d(v_1,v_{3k+3}) = k+4$

$$v_1 \to v_{3(1)+1} \to v_{3(2)+1} \to ...v_{3k+1} \to$$
$$u_{3k+1} \to u_{3k+2} \to u_{3k+3} \to v_{3k+3}.$$

Lemma 4.

Let u,v,w be three vertices on the outer cycle of $P(n,3)$, where $n=6k+4$ then

$$d(u,v)+d(v,w)+d(w,u) \leq 2d+2.$$

Proof.

By Lemma 3, $d(u_1,u_{3k+3}) = k+4 = d$.

Case(1). When k is odd.

$d\left(u_{3k+3}, u_{3[2k-(\frac{k-1}{2}-1)]+1}\right) = \frac{k+7}{2}$ and a path of length $\frac{k+7}{2}$ between u_{3k+3} and $u_{3[2k-(\frac{k-1}{2})-1]+1}$ is

$$u_{3k+3} \to v_{3(k+1)} \to v_{3(k+2)} \to v_{3(k+3)} ... \to v_{3[2k-(\frac{k-1}{2}-1)]}$$
$$\to u_{3[2k-(\frac{k-1}{2}-1)]} \to u_{3[2k-(\frac{k-1}{2}-1)]+1}$$

and

$$d\left(u_{3[2k-(\frac{k-1}{2}-1)]+1}, u_1\right) = \frac{k+5}{2}$$

because

$$u_{3[2k-(\frac{k-1}{2}-1)]+1} \to u_{3[2k-(\frac{k-1}{2})-1]+1+1} \to v_{3[2k-(\frac{k-1}{2}-1)]+1+1} \to$$
$$\to v_{3[2k-(\frac{k-1}{2}-1)]+1+1+1.3} ... \to v_{3[2k-(\frac{k-1}{2}-1)]+1+1+(\frac{k-1}{2}.3)} = v_1 \to u_1$$

Therefore,

$$d\left(u_1,u_{3k+3}\right)+d\left(u_{3k+3},u_{3[2k-(\frac{k-1}{2}-1)]+1}\right)+d\left(u_{3[2k-(\frac{k-1}{2}-1)]+1},u_1\right)$$
$$= k+4+\frac{k+7}{2}+\frac{k+5}{2} = 2d+2$$

.

Case(2). When k is even.

$$d\left(u_{3k+3}, u_{3[2k-(\frac{k}{2}-1)]+1}\right) = \frac{k+6}{2}$$

and a path of length $\frac{k+6}{2}$ between u_{3k+3} and $u_{3[2k-(\frac{k}{2}-1)]+1}$ is

$$u_{3k+3} \to v_{3(k+1)} \to v_{3(k+2)} \to v_{3(k+3)} ...$$
$$... \to v_{3[2k-(\frac{k}{2}-1)]} \to u_{3[2k-(\frac{k}{2}-1)]} \to u_{3[2k-(\frac{k}{2}-1)]+1}$$

and $d\left(u_{3[2k-(\frac{k}{2}-1)]+1}, u_1\right) = \frac{k+6}{2}$ because

$$u_{3[2k-(\frac{k}{2}-1)]+1} \to u_{3[2k-(\frac{k}{2}-1)]+1+1} \to v_{3[2k-(\frac{k}{2}-1)]+1+1} \to$$
$$\to v_{3[2k-(\frac{k}{2}-1)]+1+1+1.3} ... \to v_{3[2k-(\frac{k}{2}-1)]+1+1+(\frac{k}{2}.3)} = v_1 \to u_1$$

Therefore,

$$d(u_1,u_{3k+3})+d\left(u_{3k+3},u_{3[2k-(\frac{k}{2}-1)]+1}\right)$$

$$+d\left(u_{3[2k-(\frac{k}{2}-1)]+1},u_1\right)=k+4+\frac{k+6}{2}+\frac{k+6}{2}.$$

$$=2d+2$$

So, if u,v,w are three vertices on the outer cycle of $P(n,3)$ then

$$d(u,v)+d(v,w)+d(w,u)\leq 2d+2.$$

Lemma 5.

If u,v,w are three vertices on the inner cycles of $P(n,3)$, $n=6k+4$, then

$$d(u,v)+d(v,w)+d(w,u)\leq 2d+2.$$

Proof.

By Lemma 3, $d(v_1,v_{3k+3})=k+4=d$.

Case(1). When k is odd.

$d\left(v_{3k+3},v_{3[2k-(\frac{k-1}{2}-1)]+1}\right)=\frac{k+7}{2}$ and a path of length $\frac{k+7}{2}$ between v_{3k+3} and $v_{3[2k-(\frac{k-1}{2}-1)]+1}$ is

$$v_{3k+1}\to v_{3(k+2)}\to v_{3(k+3)}\dots\to v_{3[2k-(\frac{k-1}{2}-1)]}\to$$

$$\to u_{3[2k-(\frac{k-1}{2}-1)]}\to u_{3[2k-(\frac{k-1}{2}-1)]+1}\to v_{3[2k-(\frac{k-1}{2}-1)]+1}$$

and $d\left(v_{3[2k-(\frac{k-1}{2}-1)]+1},v_1\right)=\frac{k+5}{2}$ as

$$v_{3[2k-(\frac{k-1}{2}-1)]+1}\to u_{3[2k-(\frac{k-1}{2}-1)]+1}\to u_{3[2k-(\frac{k-1}{2}-1)]+1+1}\to v_{3[2k-(\frac{k-1}{2}-1)]+1+1}$$

$$\to v_{3[2k-(\frac{k-1}{2}-1)]+1+1+1.3}\to\dots\to v_{3[2k-(\frac{k-1}{2}-1)]+1+1+\frac{k-1}{2}.3}=v_1$$

Therefore,

$$d(v_1,v_{3k+3}),d\left(v_{3k+3},v_{3[2k-(\frac{k-1}{2}-1)]+1}\right)+d\left(v_{3[2k-(\frac{k-1}{2}-1)]+1},v_1\right)$$

$$=k+4+\frac{k+7}{2}+\frac{k+5}{2}=2d+2$$

.

Case(2). When k is even.

$d\left(v_{3k+3},v_{3[2k-(\frac{k}{2}-1)]+1}\right)=\frac{k+6}{2}$ and a path of length $\frac{k+6}{2}$ between v_{3k+3} and $v_{3[2k-(\frac{k}{2})-1]+1}$ is

$$v_3k+1\to v_{3(k+2)}\to v_{3(k+3)}\dots\to v_{3[2k-(\frac{k}{2}-1)]}\to$$

$$\to u_{3[2k-(\frac{k}{2}-1)]}\to u_{3[2k-(\frac{k}{2}-1)]+1}\to v_{3[2k-(\frac{k}{2}-1)]+1}$$

and $d\left(v_{3[2k-(\frac{k}{2}-1)]+1},v_1\right)=\frac{k+6}{2}$ as

$$v_{3[2k-(\frac{k}{2}-1)]+1}\to u_{3[2k-(\frac{k}{2}-1)]+1}\to u_{3[2k-(\frac{k}{2}-1)]+1+1}\to v_{3[2k-(\frac{k}{2}-1)]+1+1}\to$$

$$\to v_{3[2k-(\frac{k}{2}-1)]+1+1+1.3}\to\dots\to v_{3[2k-(\frac{k}{2}-1)]+1+1+\frac{k}{2}.3}=v_1$$

Therefore,

$$d(v_1,v_{3k+3}),d\left(v_{3k+3},v_{3[2k-(\frac{k}{2}-1)]+1}\right)+d\left(v_{3[2k-(\frac{k}{2}-1)]+1},v_1\right)$$

$$=k+4+\frac{k+6}{2}+\frac{k+6}{2}=2d+2$$

.

Thus if u,v,w are three vertices on the inner cycles $P(n,3)$ then

$$d(u,v)+d(v,w)+d(w,u)\leq 2d+2.$$

Lemma 6.

Let u,v,w be three vertices of $P(n,3)$, $n=6k+4$ with two vertices on the outer and one vertex on the inner cycle then

$$d(u,v)+d(v,w)+d(w,u)\leq 2d.$$

Proof.

By Lemma 3, $d(u_1,u_{3k+3})=k+4=d$.

For each vertex v_1 on the inner cycle there is only one vertex u_{3k+3} on the outer cycle at a distance $d-1$.

i.e. $d(v_1,u_{3k+3})=d-1$,

Therefore,

$$d(u_1,u_{3k+3})+d(u_{3k+3},v_1)+d(v_1,u_1)=d+(d-1)+1=2d$$

Thus if u,v,w are three vertices with two vertices on the outer and one vertex on the inner cycle of $P(n,3)$, then

$$d(u,v)+d(v,w)+d(w,u)\leq 2d.$$

We use above mentioned maximum possible sum of the pairwise distance between three vertices on the outer cycle and on the inner cycle of $P(n,3)$ together with the radio condition to determine the minimum distance between every other label (arranged in increasing order) in a radio labelling of $P(n,3)$.

Lemma 7.

Let f be radio labeling for $P(n,3)$, $n=6k+4$, where $k=2s+1, s\geq 2$.

(i) Suppose $\{x_i : 1 \le i \le n\}$ is the set of vertices on the outer cycle with labels

$f(x_1) < f(x_2) < f(x_3) < ... < f(x_{n-1} < f(x_n)$,

Then $f(x_{i+2}) - f(x_i) = f_i + f_{i+1} \ge \frac{k+5}{2}$.

(ii) Suppose $\{y_i : 1 \le i \le n\}$ is the set of vertices on the inner cycle with labels

$f(y_1) < f(y_2) < f(y_3) < ... < f(y_{n-1}) < f(y_n)$,

Then $f(y_{i+2}) - f(y_i) = f_i' + f_{i+1}' \ge \frac{k+5}{2}$.

Proof.

(i) Let $\{x_i, x_{i+1}, x_{i+2}\}$ be any set of three vertices on the outer cycle of $P(n,3)$ with $n = 6k+4$. Applying the radio condition to each pair in the vertex set $\{x_i, x_{i+1}, x_{i+2}\}$ and takes the sum of three inequalities.

$| f(x_{i+1}) - f(x_i) | \ge \text{diam}(G) - d(x_{i+1}, x_i) + 1$

$| f(x_{i+2}) - f(x_{i+1}) | \ge \text{diam}(G) - d(x_{i+2}, x_{i+1}) + 1$

$| f(x_{i+2}) - f(x_i) | \ge \text{diam}(G) - d(x_{i+2}, x_i) + 1$

$| f(x_{i+1}) - f(x_i) | + | f(x_{i+2}) - f(x_{i+1}) | + | f(x_{i+2}) - f(x_i) | \ge$

$3\text{diam}(G) - d(x_{i+1}, x_i) - d(x_{i+2}, x_{i+1}) - d(x_{i+2}, x_i) + 3$

We drop the absolute sign because $f(x_i) < f(x_{i+1}) < f(x_{i+2})$ and using Lemma 4 to obtain:

$2[f(x_{i+2}) - f(x_i)] \ge 3diam(G) - (2d+2) + 3 = d+1$

$f_i + f_{i+1} = f(x_{i+2}) - f(x_i) \ge \frac{k+5}{2}.$

(ii) Now let $\{y_i, y_{i+1}, y_{i+2}\}$ be any set of three vertices on the inner cycles of a graph $P(n,3)$ with $n = 6k+4$.

Applying radio condition to each pair in the above manner and using Lemma 5, we get

$2[f(y_{i+2}) - f(y_i)] \ge 3diam(G) - (2d+2) + 3 = d+1$

$f_i' + f_{i+1}' = [f(y_{i+2}) - f(y_i)] \ge \frac{d+1}{2} = \frac{k+5}{2}$.

The above Lemma makes it possible to calculate the minimum possible span of a radio labeling of $P(n,3)$.

Theorem 8.

For the generalized Petersen graph $P(n,3)$, $n = 6k+4$, $k = 2s+1, s \ge 2$

$$\text{rn}(P(n,3)) \ge \frac{6k^2 + 33k + 19}{2}.$$

Proof.

A generalized Petersen graph has $2n$ vertices. First divide the set of vertices into two subsets $\{u_1, u_2, u_3, ..., u_n\}$ and $\{v_1, v_2, v_3, ..., v_n\}$. Let f be a distance labeling for $P(n,3)$. We order the vertices of $P(n,3)$ on the outer cycle by $x_1, x_2, x_3, ..., x_n$ with $f(x_i) < f(x_{i+1})$ and the vertices on the inner cycle by $y_1, y_2, y_3, ..., y_n$ with $f(y_i) < f(y_{i+1})$.

Write $d = diam(P(n,3))$.

We have $d = k+4$. For $i = 1,2,3,...n-1$, set $d_i = d(x_i, x_{i+1})$ and $f_i = f(x_{i+1}) - f(x_i)$.

Then $f_i \ge d + 1 - d_i$ for all i.

By Lemma $7(i)$, the span of a distance labeling of $P(n,3)$ for the vertices on the outer cycle is

$$f(x_n) = \sum_{i=1}^{n-1} f_i = f_1 + f_2 + f_3 + ... + f_{n-2} + f_{n-1}$$

$$= \left[f(x_{2}) - f(x_{1}) \right] + \left[f(x_{3}) - f(x_{2}) \right] + ... + \left[f(x_{n-1}) - f(x_{n-2}) \right] + \left[f(x_{n}) - f(x_{n-1}) \right]$$

$$= (f_{1} + f_{2}) + (f_{3} + f_{4}) + (f_{5} + f_{6}) + ... + (f_{n-3} + f_{n-2}) + f_{n-1}$$

$$= \sum_{i=1}^{\frac{n-2}{2}} (f_{2i-1} + f_{2i}) + f_{n-1}$$

$$\ge \frac{n-2}{2}\left(\frac{k+5}{2}\right) + 1$$

Thus,

$$f(x_n) \ge \frac{3k^2 + 16k + 7}{2}$$

Applying Lemma 6 and Lemma 7(ii) to the vertices x_{n-1}, x_n, y_1 such that

$$f(x_{n-1}) < f(x_n) < f(y_1),$$

then

$$| f(y_1) - f(x_{n-1}) | \ge \frac{k+7}{2}$$

$$f(y_1) \ge f(x_{n-1}) + \frac{k+7}{2}$$

$$f(y_1) \ge \frac{3k^2 + 17k + 12}{2}$$

By Lemma $7(ii)$, the span of a distance labeling f of $P(n,3)$ for the vertices on the inner cycles is given by

$$f(y_n)-f(y_1)=\sum_{i=1}^{n-1} f_i'=(f_1'+f_2')+(f_3'+f_4)+...+(f_{n-3}'+f_{n-2}')+f_{n-1}'$$

$$=\sum_{i=1}^{\frac{n-2}{2}}(f_{2i-1}'+f_{2i}')+f_{n-1}'$$

$$\geq \frac{n-2}{2}\left(\frac{k+5}{2}\right)+1$$

Thus,

$$f(y_n)-f(y_1)\geq \frac{3k^2+16k+7}{2}$$

$$f(y_n)\geq \frac{6k^2+33k+19}{2}.$$

Lemma 9.

Let f be radio labeling for a generalized Petersen graphs, $P(n,3), n=6k+4$, where $k=2s, s\geq 3$.

(i) Suppose $\{x_i : 1\leq i\leq n\}$ is the set of vertices on the outer cycle with label

$$f(x_1)<f(x_2)<...<f(x_{n-1})<f(x_n),$$

then

$$f(x_{i+2})-f(x_i)=f(x_{i+2})-f(x_i)=f_i+f_{i+1}\geq` \frac{k+4}{2}+\frac{1}{2}p= \frac{k+6}{2}$$

.

(ii) Suppose $\{y_i : 1\leq i\leq n\}$ is the set of vertices on the inner cycles with label

$$f(y_1)<f(y_2)<...<f(y_{n-1})<f(y_n),$$

then $f(y_{i+2})-f(y_i)=f_i'+f_{i+1}'\geq` \frac{k+6}{2}+\frac{1}{2}p= \frac{k+6}{2}$.

Proof.

Proof is similar to Lemma 7.

Theorem 10.

For the generalized Petersen graphs, $P(n,3)$, $n=6k+4$ and $k=2s$ where $s\geq 3$

$$\text{rn}(P(n,3))\geq \frac{6k^2+39k+22}{2}.$$

Proof.

A generalized Petersen graph has $2n$ vertices. First divide the set of vertices into two subsets $u_1,u_2,u_3,...,u_n$ and $v_1,v_2,v_3,...,v_n$. Let f be a distance labeling for $P(n,3)$. We order the vertices of $P(n,3)$ on the outer cycle by $x_1,x_2,x_3,...,x_n$ with $f(x_i)<f(x_{i+1})$ and the vertices on the inner cycles by $y_1,y_2,y_3,...,y_n$ with $f(y_i)<f(y_{i+1})$.

Denote $d=diam(P(n,3))$. Then $d=k+4$.

For $i=1,2,3,...n-1$, set $d_i=d(x_i,x_{i+1})$

and $f_i=f(x_{i+1})-f(x_i)$

Then $f_i\geq d+1-d_i$ for all i.

By Lemma $9(i)$, the span of a distance labeling f of $P(n,3)$ for the vertices on the outer cycle is:

$$f(x_n)=\sum_{i=1}^{n-1} f_i=f_1+f_2+f_3+....+f_{n-2}+f_{n-1}$$

$$=\left[f(x_{2})-f(x_{1})\right]+\left[f(x_{3})-f(x_{2})\right]+\ ...\ +\left[f(x_{n-1})-f(x_{n-2})\right]+\left[f(x_{n})-f(x_{n-1})\right]$$

$$=(f_{1}+f_{2})+(f_{3}+f_{4})+(f_{5}+f_{6})+\ ...\ +(f_{n-3}+f_{n-2})+f_{n-1}$$

$$=\sum_{i=1}^{\frac{n-2}{2}}(f_{2i-1}+f_{2i})+f_{n-1}$$

$$\geq \frac{n-2}{2}\left(\frac{k+6}{2}\right)+1$$

Thus,

$$f(x_n)\geq \frac{3k^2+19k+8}{2}$$

Applying Lemma 6 and Lemma 9(ii) to the vertices x_{n-1},x_n,y_1 such that

$$f(x_{n-1})<f(x_n)<f(y_1),$$

then

$$|f(y_1)-f(x_{n-1})|\geq \frac{k+8}{2}$$

$$f(y_1)\geq f(x_{n-1})+\frac{k+8}{2}-1$$

$$f(y_1)\geq \frac{3k^2+20k+14}{2}$$

By Lemma $9(ii)$, the span of distance labeling of $P(n,3)$ for the vertices on the inner cycles is

$$f(y_n)-f(y_1)=\sum_{i=1}^{n-1} f_i'=(f_1'+f_2')+(f_3'+f_4)+...+(f_{n-3}'+f_{n-2}')+f_{n-1}'$$

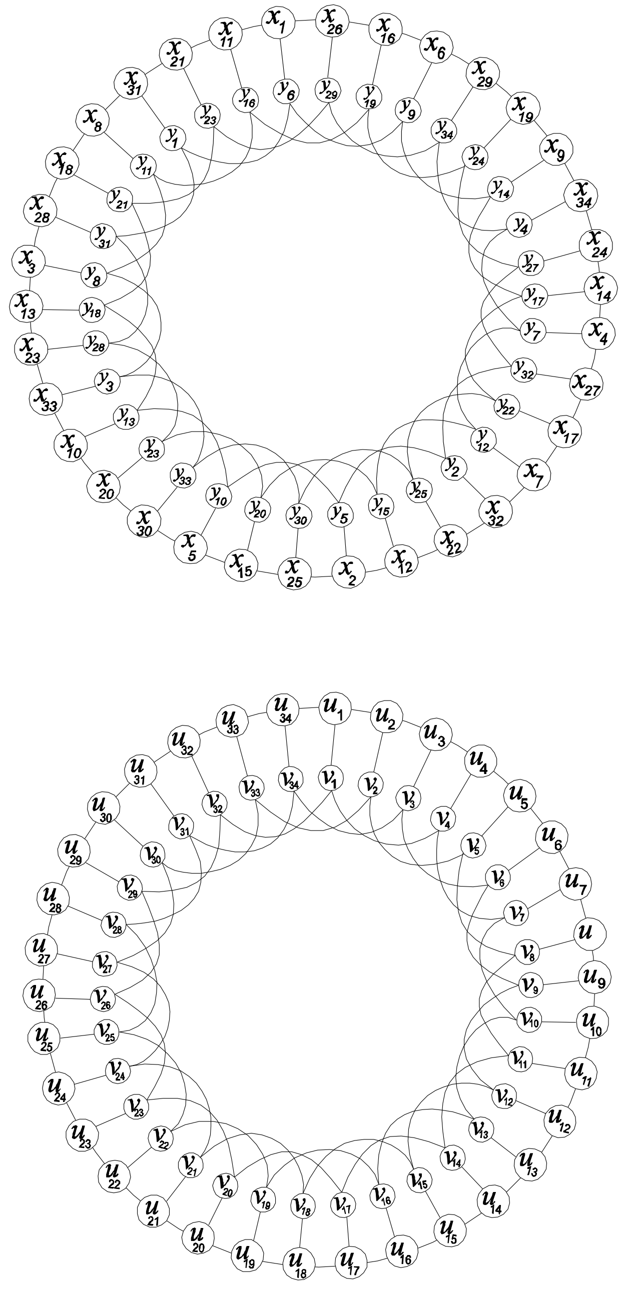
$$=\sum_{i=1}^{\frac{n-2}{2}}(f_{2i-1}'+f_{2i}')+f_{n-1}'$$

$$\geq \frac{n-2}{2}\left(\frac{k+6}{2}\right)+1$$

Therefore,

$$f(y_n)-f(y_1)\geq\frac{3k^2+19k+8}{2}$$

$$f(y_n)\geq\frac{6k^2+39k+22}{2}.$$



Ordinary labeling and radio labeling for $P(34,3)$.

4. *AN UPPER BOUND FOR* $P(n,3)$

To complete the proof of Theorem 1 and Theorem 2 it remains to find the radio labeling for $P(n,3)$ with span equal to the desired number.

The labeling is generated by pair of three sequences,

the distance gap sequences

$$D=(d_1,d_2,d_3,....,d_{n-1})$$

$$D'=(d_1',d_2',d_3',...,d_{n-1}')$$

the color gap sequences

$$F=(f_1,f_2,f_3,....,f_{n-1})$$

$$F'=(f_1',f_2',f_3',...,f_{n-1}')$$

and the vertex gap sequences T and T'

$$T=(t_1,t_2,t_3,....,t_{n-1})$$

$$T'=(t_1',t_2',t_3',...,t_{n-1}')$$

Case 1. When k is odd.

The distance gap sequences are given by:

$$d_i=d_i'=\begin{cases}k+4, & if\ \ i\ \ is\ \ odd;\\ \frac{k+7}{2}, & if\ \ i\ \ is\ \ even.\end{cases}$$

For $i=1,2,3,...,n-1$, $d_i'=d(y_i,y_{i+1})$ and

$$d'=d(x_n,y_1)=\frac{k+5}{2}.$$

The color gap sequences F and F' are given by:

$$f_i=f_i'=\begin{cases}1, & if\ \ i\ \ is\ \ odd;\\ \frac{k+3}{2}, & if\ \ i\ \ is\ \ even.\end{cases}$$

$$f'=f(y_1)-f(x_n)=\frac{k+5}{2}.$$

The vertex gap sequences are:

$$t_i=t_i'=\begin{cases}3k+1, & if\ \ i\ \ is\ \ odd;\\ \frac{3k+3}{2}, & if\ \ i\ \ is\ \ even.\end{cases}$$

Where t_i denotes number of vertices between x_i and x_{i+1} on the outer cycle and t_i' denotes number of vertices between y_i and y_{i+1} on the inner cycles.

Let $\pi,\pi':\{1,2,3,...,n\}\rightarrow\{1,2,3,...,n\}$ be defined by $\pi(1)=1$

and

$$\pi'(1)=\begin{cases}1, & if\ \ k\equiv 3(\mathrm{mod}4)\\ 6k+2, & if\ \ k\equiv 1(\mathrm{mod}4).\end{cases}$$

$$\pi(i+1)=\pi(i)+t_i+1(modn)$$

$$\pi'(i+1)=\pi'(i)+t_i'+1(modn)$$

Assuming $x_i = u_{\pi(i)}$ and $y_i = v_{\pi'(i)}$ for $i = 1,2,3,...,n$.

Then $x_1, x_2, x_3, ..., x_n$ is an ordering of the vertices of $P(n,3)$ on the outer cycle and $y_1, y_2, y_3, ..., y_n$ is an ordering of the vertices of $P(n,3)$ on the inner cycles.

Assume $f(x_1) = 0$, and $f(x_{i+1}) = f(x_i) + f_i$.

Then for $i = 1,2,3,...,3k+2$

$$\pi(2i-1) = (i-1)(3k+1) + (i-1)\left(\frac{3k+3}{2}\right) + 2(i-1) + 1 (\text{mod}\, n)$$

$$\pi(2i) = i(3k+1) + (i-1)\left(\frac{3k+3}{2}\right) + (2i-1) + 1 (\text{mod}\, n).$$

Case (i): When $k \equiv 3(\text{mod}\,4)$. Then for

$i = 1,2,3,...,3k+2,$

$\pi'(2i-1) = \pi(2i-1),$

$\pi'(2i) = \pi(2i).$

Case (ii): When $k \equiv 1(\text{mod}\,4)$. Then for

$i = 1,2,3,...,3k+2,$

$\pi'(2i-1) = 6k+2+\pi(2i-1),$

$\pi'(2i) = 6k+2+\pi(2i).$

We will show that each of the sequences given above, the corresponding π, π' are permutations.

For this it is sufficient to show that π is a permutation.

Note that g.c.d $(n,k) = 1$ and $3k+3 \equiv -3k-1(\text{mod}\, n)$. Thus,

$$(3k+3)(i-i') \equiv (3k+1)(i-i') \not\equiv 0(\text{mod}\, n) \;\; when \;\; 0 < i-i' < \frac{n}{2}.$$

This implies that $\pi(2i) \neq \pi(2i')$ or

$\pi(2i-1) \neq \pi(2i'-1)$ for $i \neq i'$.

However, if $\pi(2i) = \pi(2i'-1)$, then we get

$3.(3k+3)(i-i') \equiv -(3k+4)(\text{mod}\, n)$

$3.(3k+1)(i'-i) \equiv (3k+2)(\text{mod}\, n)$

$6.(3k+1)(i'-i) \equiv (6k+4)(\text{mod}\, n)$

$6.(3k+1)(i-i') \equiv 0(\text{mod}\, n)$

Since $\text{g.c.d.}(3k+1,\ n) = 2$, it follows that $i - i' \equiv 0(\text{mod}\, n)$. But this is not possible because

$$0 < i-i' < \frac{n}{2} = 3k+2.$$

Thus, π is a permutation consequently π' is also.

The span of f is equal to:

$$\text{f}\_\{1\} + \text{f}\_\{2\} + \text{f}\_\{3\} +, \ldots, \text{f}\_\{n-2\} + \text{f}\_\{n-1\} + \text{f}' +$$
$$\text{f}'\_\{1\} + \text{f}'\_\{2\} + \text{f}'\_\{3\} +, \ldots, \text{f}'\_\{n-2\} + \text{f}'\_\{n-1\}$$
$$= \left[\left(\text{f}\_\{1\} + \text{f}\_\{3\} + \text{f}\_\{5\} +, \ldots, + \text{f}\_\{n-1\}\right)\right] +$$
$$\left[\left(\text{f}\_\{2\} + \text{f}\_\{4\} + \text{f}\_\{6\} +, \ldots, + \text{f}\_\{n-2\}\right)\right] + \text{f}'$$
$$+ \left[\left(\text{f}'\_\{1\} + \text{f}'\_\{3\} + \text{f}'\_\{5\} +, \ldots, + \text{f}'\_\{n-1\}\right)\right] +$$
$$\left[\left(\text{f}'\_\{2\} + \text{f}'\_\{4\} + \text{f}'\_\{6\} +, \ldots, + \text{f}'\_\{n-2\}\right)\right]$$
$$= \frac{n}{2}(1) + \frac{n-2}{2}\left(\frac{k+3}{2}\right) + \frac{k+5}{2} + 2 + \frac{n}{2}(1) + \frac{n-2}{2}\left(\frac{k+3}{2}\right)$$
$$= \frac{6k^2 + 33k + 19}{2}$$

Case 2. When k is even.

Case (i). For $k \equiv 0(\text{mod}\,4)$, $k = 4s$ where $s \geq 2$.

The distance gap sequences D and D' are given by:

$$d_i = \begin{cases} k+4, & if \;\; i \;\; is \;\; odd; \\ \frac{k}{2}+3, & if \;\; i \;\; is \;\; even. \end{cases}$$

$$d'_i = \begin{cases} k+4, & if \;\; i \;\; is \;\; odd; \\ \frac{k}{2}+4, & if \;\; i \;\; is \;\; even. \end{cases}$$

Where $d'_i = d(y_i, y_{i+1})$ for $i = 1,2,3,...,n-1$ and

$$d' = d(x_n, y_1) = \frac{k}{2} + 2.$$

The color gap sequences F and F' are given by:

$$f_i = f'_i = \begin{cases} 1, & if \;\; i \;\; is \;\; odd; \\ \frac{k}{2}+2, & if \;\; i \;\; is \;\; even. \end{cases}$$

$$f' = \frac{k}{2} + 3.$$

The vertex gap sequences are:

$$t_i = \begin{cases} 3k+1, & if \;\; i \;\; is \;\; odd; \\ \frac{3k}{2}, & if \;\; i \equiv 2(\text{mod}\,4); \\ \frac{3k}{2}+2, & if \;\; i \equiv 0(\text{mod}\,4). \end{cases}$$

$$t_i' = \begin{cases} 3k+1, & if \quad i \quad is \quad odd; \\ \frac{3k}{2}, & if \quad i \equiv 2(\bmod 4); \\ \frac{3k}{2}+3, & if \quad i \equiv 0(\bmod 4). \end{cases}$$

where t_i and t_i' denotes the number of vertices on outer cycle and inner cycles.

Let $\theta, \theta' : \{1,2,3,...,n\} \to \{1,2,3,...,n\}$ be defined by $\theta(1)=1 \quad and \quad \theta'(1)=1$.

$$\theta(i+1) = \theta(i) + t_i + 1(\bmod n)$$

$$\theta'(i+1) = \theta'(i) + t_i' + 1(\bmod n)$$

Assuming $x_i = u_{\theta(i)}$ and $y_i = v_{\theta'(i)}$ for $i = 1,2,3,...,n$.

Then $x_1, x_2, x_3, ..., x_n$ is an ordering of the vertices of $P(n,3)$ on the outer cycle and $y_1, y_2, y_3, ..., y_n$ is an ordering of the vertices of $P(n,3)$ on the inner cycles.

Let $f(x_1)=0$, $f(x_{i+1}) = f(x_i) + f_i$.

Then for $i = 1,2,3,...,\frac{3k}{2}+1$,

$$\theta(4i-2) = (2i-1)(3k+1) + (2i-2)\left(\frac{3k}{2}\right) + (6i-5) + 1(\bmod n)$$

$$\theta(4i-1) = (2i-1)(3k+1) + (2i-1)\left(\frac{3k}{2}\right) + (6i-4) + 1(\bmod n)$$

$$\theta(4i) = 2i(3k+1) + (2i-1)\left(\frac{3k}{2}\right) + (6i-3) + 1(\bmod n)$$

and

$$\theta(4i+1) = 2i(3k+1) + 2i\left(\frac{3k}{2}\right) + 6i + 1(\bmod n), \ for \quad i = 0,1,2,...,\frac{3k}{2}.$$

$$\theta'(4i-2) = (2i-1)(3k+1) + (2i-2)\left(\frac{3k}{2}\right) + (7i-6) + 1(\bmod n)$$

$$= \theta(4i-2) + i - 1$$

$$\theta'(4i-1) = (2i-1)(3k+1) + (2i-1)\left(\frac{3k}{2}\right) + (7i-5) + 1(\bmod n)$$

$$= \theta(4i-1) + i - 1$$

$$\theta'(4i) = 2i(3k+1) + (2i-1)\left(\frac{3k}{2}\right) + (7i-4) + 1(\bmod n)$$

$$= \theta(4i) + i - 1$$

and

$$\theta'(4i+1) = 2i(3k+1) + 2i\left(\frac{3k}{2}\right) + 7i + 1(\bmod n)$$

$$= \theta(4i+1) + i, \text{ for } i = 0,1,2,...,\frac{3k}{2}.$$

Case (ii):

For $k \equiv 2(\bmod 4)$, i.e $k = 4s+2$ where $s \geq 1$

The distance gap sequences are given by:

$$d_i = \begin{cases} k+4, & if \quad i \quad is \quad odd; \\ \frac{k}{2}+3, & if \quad i \quad is \quad even. \end{cases}$$

$$d_i' = \begin{cases} k+4, & if \quad i \quad is \quad odd; \\ \frac{k}{2}+3, & if \quad i \equiv 2(\bmod 4); \\ \frac{k}{2}+4, & if \quad i \equiv 0(\bmod 4). \end{cases}$$

For $i = 1,2,3,...,n-1$, $d_i' = d(y_i, y_{i+1})$ and

$$d' = d(x_n, y_1) = \frac{k}{2}+2,$$

$$d'' = d(x_{3k+2}, x_{3k+3}) = \frac{k}{2}+4.$$

The color gap sequences F and F' are given by:

$$f_i = f_i' = \begin{cases} 1, & if \quad i \quad is \quad odd; \\ \frac{k}{2}+2, & if \quad i \quad is \quad even. \end{cases}$$

$$f' = f(y_1) - f(x_n) = \frac{k}{2}+3.$$

The vertex gap sequences are:

$$t_i = \begin{cases} 3k+1, & if \quad i \quad is \quad odd; \\ \frac{3k}{2}, & if \quad i \equiv 2(\bmod 4); \\ \frac{3k}{2}+2, & if \quad i \equiv 0(\bmod 4). \end{cases}$$

$$t_{3k+2} = \frac{3k}{2}+2.$$

$$t_i' = \begin{cases} 3k+1, & if \quad i \quad is \quad odd; \\ \frac{3k}{2}, & if \quad i \equiv 2(\bmod 4); \\ \frac{3k}{2}+3, & if \quad i \equiv 0(\bmod 4). \end{cases}$$

where t_i and t_i' denotes the number of vertices on the outer cycle and inner cycle respectively.

Let $\phi,\phi':\{1,2,3,\dots,n\}\to\{1,2,3,\dots,n\}$ be defined by $\phi(1)=1$ and $\phi'(1)=2$

$$\phi(i+1)=\phi(i)+t_i+1(mod\,n)$$

$$\phi'(i+1)=\phi'(i)+t_i'+1(mod\,n)$$

Then for $i=1,2,\dots,\frac{3k+2}{4}$,

$$\phi(4i-2)=\theta(4i-2)$$

$$\phi(4i-1)=\theta(4i-1)$$

$$\phi(4i)=\theta(4i)$$

and for $i=0,1,2,\dots,\frac{3k-2}{4}$,

$$\phi(4i+1)=\theta(4i+1).$$

Now, for $i=\frac{3k+6}{4},\dots,\frac{6k+4}{4}$,

$$\phi(4i-2)=\frac{3k}{2}+3+\theta(4i-2)$$

$$\phi(4i-1)=\frac{3k}{2}+3+\theta(4i-1)$$

$$\phi(4i)=\frac{3k}{2}+3+\theta(4i)$$

and for $i=\frac{3k+2}{4},\dots,\frac{6k}{4}$,

$$\phi(4i+1)=\frac{3k}{2}+3+\theta(4i+1).$$

Then for $i=1,2,3,\dots,\frac{3k}{2}+1$

$$\phi'(4i-2)=\theta'(4i-2)+1$$
$$\phi'(4i-1)=\theta'(4i-1)+1$$

$$\phi'(4i)=\theta'(4i)+1$$

and

for $i=0,1,2,\dots,\frac{3k}{2}$, $\phi'(4i+1)=\theta'(4i+1)+1$

We will show that each of the sequence given above the corresponding θ, θ', ϕ and ϕ' are permutations. For this it is sufficient to show that θ is a permutation.

Since k is even, so

$\text{g.c.d.}(3k+4,n)=2$ and $9k+8\equiv 3k+4(\text{mod}\,n)$. Thus,

$$(9k+8)(i-i')\equiv(3k+4)(i-i')\not\equiv 0(\text{mod}\,n)$$

when $0<i-i'<\frac{n}{4}=\frac{3k}{2}+1$.

If $(3k+4)(i-i')\equiv 0(\text{mod}\,n)$ then $i-i'\equiv 0(\text{mod}\frac{n}{2})$. It means that $\frac{n}{2}$ *divides* $i-i'$ which is a contradiction to the fact that $i-i'<\frac{n}{4}$.

This implies that $\theta(4i)\neq\theta(4i')$ for $i\neq i'$.

Similarly, it can be easily show that for $i=1,2,\dots,\frac{3k}{2}+1$ and $i\neq i'$,

$$\theta(4i-2)\neq\theta(4i'-2)$$

$$\theta(4i-1)\neq\theta(4i'-1)$$

And $\theta(4i+1)\neq\theta(4i'+1)$, for $i=0,1,2,\dots,\frac{3k}{2}$.

However, if $\theta(4i)=\theta(4i'-1)$ then we have

$$(9k+8)(i-i')\equiv-(3k+2)(\text{mod}\,n)$$

$$(3k+4)(i'-i)\equiv(3k+2)(\text{mod}\,n)$$

$$2(3k+4)(i'-i)\equiv 0(\text{mod}\,n)$$

$$2(3k+4)(i'-i)\equiv 0.2(\text{mod}\,n)$$

Since

$\text{g.c.d.}(2(3k+4),n)=2$ therefore $i'-i\equiv 0(\text{mod}\frac{n}{2})$, it follows that $\frac{n}{2}<i'-i<\frac{n}{4}$. This is not possible.

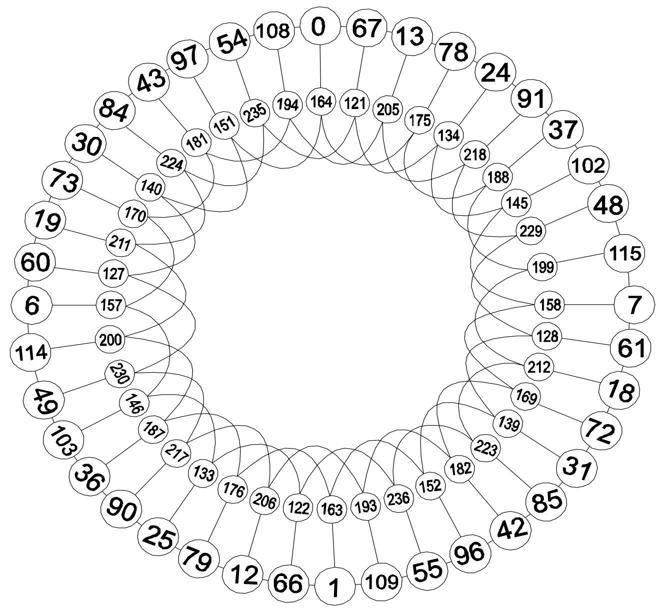
The remaining cases shown are on the similar way. Thus θ is a permutation and consequently θ', ϕ and ϕ' are also permutations

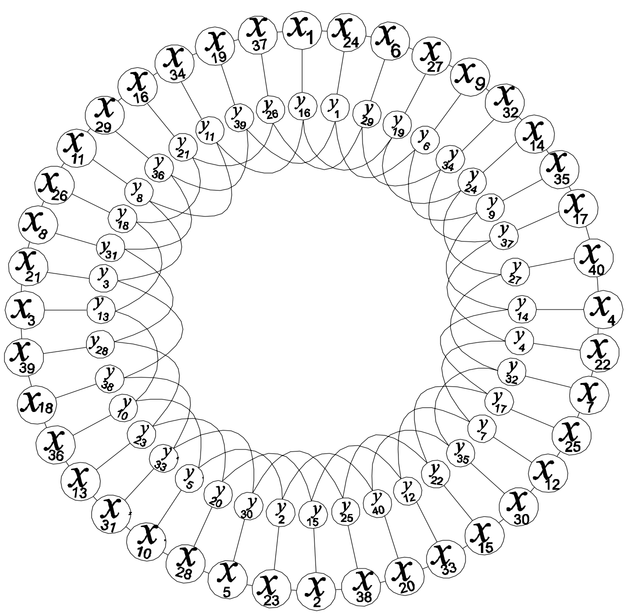
The span of f is equal to

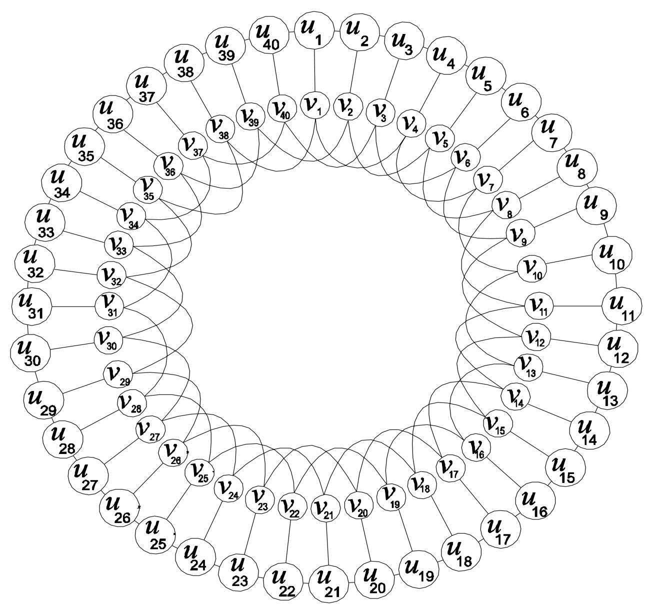
$$f\_\{1\}+f\_\{2\}+f\_\{3\}+,\ \dots\ ,f\_\{n-2\}+f\_\{n-1\}+f'+$$
$$f'\_\{1\}+f'\_\{2\}+f'\_\{3\}+,\ \dots\ ,f'\_\{n-2\}+f'\_\{n-1\}$$

$$=\left[\left(f\_\{1\}+f\_\{3\}+f\_\{5\}+,\ \ldots\ ,+f\_\{n-1\}\right)\right]$$
$$+\left[\left(f\_\{2\}+f\_\{4\}+f\_\{6\}+,\ \ldots\ ,+f\_\{n-2\}\right)\right]$$
$$+f'+\left[\left(f'\_\{1\}+f'\_\{3\}+,\ \ldots\ ,+f'\_\{n-1\}\right)\right]$$
$$+\left[\left(f'\_\{2\}+f'\_\{4\}++,\ \ldots\ ,+f'\_\{n-2\}\right)\right]$$
$$=\left[\left(f\_\{1\}+f\_\{3\}+,\ \ldots\ ,+f\_\{n-1\}\right)\right]$$
$$+\left[\left(f\_\{2\}+f\_\{6\}+,\ \ldots\ ,+f\_\{n-4\}\right)\right]$$
$$+\left[\left(f\_\{4\}+f\_\{8\}+,\ \ldots\ ,+f\_\{n-2\}\right)\right]$$
$$+f'+\left[\left(f'\_\{1\}+f'\_\{3\}+,\ \ldots\ ,+f'\_\{n-1\}\right)\right]$$
$$+\left[\left(f'\_\{2\}+f'\_\{4\}+,\ \ldots\ ,+f'\_\{n-2\}\right)\right]$$
$$=\frac{n}{2}(1)+\frac{n-2}{2}\left(\frac{k+4}{2}\right)+\left(\frac{k+6}{2}\right)+\frac{n}{2}(1)+\frac{n-2}{2}\left(\frac{k+4}{2}\right)$$

$$=\frac{6k^2+39k+22}{2}.$$







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