GENERALIZED CHARACTERISTIC SETS FOR ORDINARY DIFFERENTIAL POLYNOMIAL SETS

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ABSTRACT. *Differential polynomial sets and systems can be triangularized by using Ritt-Wu's method of characteristic sets, which uses pseudo- division to eliminate variables successively. Jin et al showed that pseudo-division may be replaced by other admissible reductions to compute generalized characteristic sets in the algebraic case. In this paper we extend the work of Jin et al to the ordinary differential case and present an algorithmic scheme for the computation of generalized differential characteristic sets of ordinary differential polynomial sets. The scheme is implemented with some specific admissible differential reductions. Preliminary results of experiments in Maple show that the algorithm using our scheme performs often better than the characteristic set algorithm based on pseudo-division in terms of efficiency and simplicity of output.*

Key Words: Algorithmic scheme, differential polynomial, characteristic set, pseudo-division, characteristic set, admissible reduction

1. INTRODUCTION

The method of characteristic sets developed initially by Ritt [16] and Wu [22] has become a standard tool for the study of systems of polynomial and algebraic differential equations. By means of constructing characteristic sets, an arbitrary system of polynomials or differential polynomials can be triangularized, or be decomposed into special systems of triangular form. Following the seminal work of Wu [22], extensive investigations have been carried out by many researchers in the last three decades (see [1,5,6,10,14,15,18,19,24,26] for example), which makes the characteristic set method more and more popular and powerful in terms of computational complexity, theoretical generality, and practical applicability.

The method has been well studied for the algebraic case, with efficient implementations [6,7,15,17,25] and remarkable applications [3,5,9,11,12,13,14,18,19,26,27], in the differential case there are naturally more difficult issues which need be addressed. One of the key issues in the computation of characteristic sets is the expression swell of immediate polynomials in the process of variable elimination, which is performed by means of pseudo-division, the main operation used in Ritt-Wu's characteristic set algorithm.

Wang [6] refined Ritt-Wu's algorithm by using one-step pseudo-reduction (instead of pseudo-division) with strategies for the selection of reductends and optimal reductors. He concluded that optimal selection of reductors as well as heuristic generation of s-polynomials can help speed up the computation of characteristic sets, sometimes considerably, resulting in simple output for large problems. Recall that for C to be a characteristic set of a polynomial set P , a necessary condition is that all the polynomials in P have pseudo-remainder 0 with respect to C . In order to have the concept of characteristic set generalized, Jin, Li and Wang [17] weakened the necessary condition by replacing P with an arbitrary polynomial set that generates the same ideal as P . With this weakening and following the work [6], they proposed a general algorithmic scheme for computing generalized characteristic sets by using admissible reductions other than pseudo-division.

In this paper we extend the work of Jin and others [17] to the ordinary differential case and present a similar algorithmic scheme for the computation of generalized differential characteristic sets of ordinary differential polynomial sets. We introduce four specific admissible differential reductions and discuss the implementation of the scheme with these reductions. Preliminary results of experiments in Maple show that the algorithm using our scheme performs often better than the characteristic set algorithm based on pseudo-division in terms of efficiency and simplicity of output.

The paper is structured as follows. After some necessary notations and the basic terminology for differential polynomials sets, we give a brief review on ordinary differential characteristic sets and explain how to compute them in Section 2. In Section 3, few admissible reductions for ordinary differential polynomial sets are introduced, their correctness is demonstrated and we define generalized characteristic sets. In section 4 we describe our main algorithm NewdCharSet, together with its sub-algorithm dMedSet, for the computation of generalized characteristic sets. A sample sub-algorithm dFind3R for finding optimal triples of d-reductends, d-reductors and admissible d-reductions is formulated. The correctness of the described algorithms is proved. Finally in the Section 5 some test examples and experimental results are given.

2. PRELIMINARIES

A. Terminology and notations

We recall some concepts which are used in this paper and on which more details can be found in [5], [16], [22].

A differential field (d-field) is a field together with a third(unary) operation "$'$", called the differential operation satisfying the properties given below:

$$(a+b)' = a' + b', (ab)' = a'b + ab'$$

Let $\mathcal{K}$ be an ordinary d-field of characteristic 0, which consists of functions of a variable t and $x_1 \prec x_2 \prec \cdots x_n$ be aset of$n$ variables, abbreviated as x. The $j^{th} (j \geq 0)$ derivate of x_i is denoted by x_{ij}. Thus we have $x_i = x_{i0,} (x_i)' = x_{i1}$, etc. By an ordinary *differential polynomial*, or d-polynomial for short, we mean a polynomial in $x_1, x_2, \dots, x_n$ and their derivatives $x_{ij}$with coefficients in $\mathcal{K}$ (e.g $Q(t)$, the field of rational functions of t, where Q stands for the field of rational numbers). The set of all such ordinary d-polynomials if denoted by $\mathcal{K}\{x_1, x_2, \dots, x_n\}$ or $\mathcal{K}\{x\}$ for short.

Let $P \in \mathcal{K}\{x_1, x_2, \dots, x_n\}$ be a non-zero d-polynomial, denote the *degree* and the *leading coefficient* of P with respect to (w.r.t) the variable x_{iq} by $\deg(P, x_{iq})$ and lc (P, x_{iq}) respectively. We denote the class of P by cls(P) which is the largest p such that some x_{pi} actually occurs in P. If $P \in \mathcal{K}$, the $\text{cls}(P) = 0$.

The *order* of P w.r.t. x_i, denoted by ord (P, x_i), is the greatest j if exists, such that $\deg(P, x_{ij}) > 0$. If $\deg(P, x_{ij}) = 0$ for any $j \geq 0$, then define order of P w.r.t. x_i to be 1, i.e., $\text{ord}(P, x_i) = -1$. Let P be a d-polynomial with $q =\text{ord}(P, x_i)$ and $d =\deg(P, x_{iq})$; then pair $\langle q, d\rangle$ is called the rank of P w.r.t. x_i and denoted by rank (P, x_i). We order $\langle q, d\rangle \prec \langle q', d'\rangle$ if $q < q'$ or $q = q'$, $d < d'$.

Fix the variables ordering as $t \prec x_1 \prec x_2 \prec \cdots \prec x_n$ and order $x_{ij \prec} x_{ik}$ if $j < k$. The *leading variable* of P, denoted by lvar(P), is defied to be x_p with the biggest index p such that deg $(P, x_{pj}) > 0$ for some j if $P \notin \mathcal{K}$, or t otherwise.

Let P be a non-zero d-polynomial with $\text{cls}(P) = p > 0$ and $\text{rank}(P, x_p) = \langle q, d\rangle$. Then P can be written in the following form:

$$P = P_0 x_{pq}^d + P_1 x_{pq}^{d-1} + \cdots + P_d \ (P_0 \neq 0)$$

where $\text{ord}(P_i, x_p) < q$ for each i. We call P_0 the *initial* of P, denoted by $\text{ini}(P)$ and x_{pq} the *lead* of P, denoted by $\text{lead}(P)$. Moreover, define $\text{ld}(P) \coloneqq \text{ld}(P, \text{lead}(P))$ and $\text{ini}(P)$, the initial of P to be the leading coefficient of P w.r.t. the lead of P, i.e., $\text{ini}(P) \coloneqq \text{lc}(P, \text{lead}(P))$. The d-polynomial $P_1 x_{pq}^{d-1} + \cdots + P_d$ is called the *reductum* of P, denoted by $\text{red}(P)$. The d-polynomial $\frac{\partial P}{\partial x_{pq}}$ is known as the *separant* of P and denoted by $\text{sep}(P)$. Let Q be any other d-polynomial. Pseduo-divining Q by P and its derivates in x_p, we get

$$\text{sep}^{\alpha}(P)\text{ini}(P)^{\beta} Q = A_1 \frac{d^{k_1}P}{dt^{k_1}} + \cdots + A_s \frac{d^{k_s}P}{dt^{k_s}} + R$$

where α, β, k_j are non-negative integers and R is a d-polynomial with the condition $\text{rank}(R, x_p) \prec \langle q, d\rangle$ (for details the reader may see [16]). R is called the d-*pseudo remainder* of Q w.r.t. P and denote it by $\text{d-prem}(Q, P)$, where R is not necessarily unique. We put $\text{d-prem}(Q, P) \coloneqq 0$ when d-prem $P \neq 0$ and $\text{d-premcls}(P) = 0$.

A *differential polynomial set* (d-polynomial set) is a finite set of non-zero d-polynomials and a *differential polynomial system* (d-polynomial system) is a pair of d-polynomial sets. Let $\widetilde{\mathcal{K}}$ denote the algebraic closure of $\mathcal{K}$. A common *d-zero* of two d-polynomials P and Q is defined to be a common zero of all linear combinations of derivatives of P and Q.

Let $\mathbb{P}$ and $\mathbb{Q}$ be two d-polynomial sets, we denote by Zero $(\mathbb{P}/\mathbb{Q})$, the set of all common d-zeros (in $\widetilde{\mathcal{K}}$) of the d-polynomial in $\mathbb{P}$ which are not d-zero of any d-polynomial in $\mathbb{Q}$. That is $\text{Zero}(\mathbb{P}/\mathbb{Q}) = \{x \in \widetilde{\mathcal{K}}^n | P(x) = 0, Q(x) \neq 0, \forall P \in \mathbb{P}, Q \in \mathbb{Q}\}$. By a d-polynomial system, we mean a pair $[\mathbb{P}, \mathbb{Q}]$ of d-polynomial set with $\mathbb{Q}$ possibly empty. A d-Zero of $[\mathbb{P}, \mathbb{Q}]$ is meant an element of $\text{Zero}(\mathbb{P}/\mathbb{Q})$.

B. Ordinary differential characteristic sets

The *characteristic set* (char set) method of solving polynomial equations is naturally extended to the differential case which gives rise to an algorithmic method of solving an arbitrary set of differential polynomials. The idea of the method is reducing the set of d-polynomials in general form to the set of d-polynomials in the d-triangular form. With the help of this method, solving a set of d-polynomials can be reduced to solving a univariate set of d-polynomials in the cascaded form.

The *differential characteristic* (d-char) set method can also be applied to computation of the dimension, the degree, and the order of a set of d-polynomials, resolution of the radical ideal membership problem, and proof of theorems from elementary and differential geometries [22].

For defining the *partial ordering* of d-polynomial sets let us consider at first that such d-polynomial sets are well arranged in the following sense. The d-polynomials present in the set are non-constant ones and so are arranged with the classes c_i steadily increasing i.e.,$0 < c_1 < \cdots < c_r$. The leading coefficient or the initial of the *i*-th d-polynomial in the set is either a non-zero constant or has a class less that c_i. It means if it is of class c_j, $1 < j < i$, it should have a degree less than that of *j*-th d-polynomial in the set, such a d-polynomial set is called the d-*ascending set*. Thus some partial ordering is introduced among set of all such d-ascending sets, with the set which consists of a single non-zero constant, is considered, as the *trivial d-ascending set* to be arranged in the lowest ordering.

Consider any finite set of the non-zero d-polynomials. For such a d-polynomial set, any d-ascending set of the lowest order contained in the given set is called the *d-basic set*. A partial ordering is unambiguously introduced among all non-empty d-polynomial sets according to the partial ordering of their d-basic sets. Any d-polynomial set containing a non-zero constant d-polynomial will clearly be lowest ordering. Now consider such a given set $\mathbb{P}$ and consider the scheme below:

$$\begin{array}{ccccccccc} \mathbb{P} & = & \mathbb{P}_0 & \dots & \mathbb{P}_i & \dots & \mathbb{P}_m & & \\ & & \mathbb{B}_0 & \mathbb{B}_1 & \dots & \mathbb{B}_i & \dots & \mathbb{B}_m = & \mathbb{C} \\ & & \mathbb{R}_0 & \mathbb{R}_1 & \dots & \mathbb{R}_i & \dots & \mathbb{R}_m = & \emptyset \end{array}$$

In this scheme, each $\mathbb{B}_i$ is the d-basic set of $\mathbb{P}_i$. Each $\mathbb{R}_i$ is the set of non-zero d-remainders of d-polynomials $\mathbb{P}_i \backslash \mathbb{B}_i$ w.r.t. $\mathbb{B}_i$. If $\mathbb{R}_i$ is non-empty then $\mathbb{P}_{i+1} = \mathbb{P} \cup \mathbb{B}_i \cup \mathbb{R}_i$. It is easy to prove that the sequence of $\mathbb{B}_i$ is steadily decreasing sequence, i.e., $\mathbb{B}_0 \succ \mathbb{B}_1 \succ \ldots \succ \mathbb{B}_r \succ \cdots$. Such a sequence cannot be infinite and it should be terminate at some stage m with $\mathbb{R} = \emptyset$. The corresponding d-basic set $\mathbb{B}_m = \mathbb{C}$ is then called the d-char set of $\mathbb{P}$.

Lemma 2.1: Let $\mathbb{P} \subset \mathcal{K}\{x\}$ be a non-empty d-polynomial set having a d-basic set $\mathbb{B} = \{B_1, B_2, \dots, B_r\}$, where $\text{cls}(B_1) > 0$. If B is a non-zero d-polynomial reduced w.r.t. $\mathbb{B}$, then $\mathbb{P} \cup \{B\}$ has a d-basic set of rank lower than that of $\mathbb{B}$.

Proof: Let $\mathbb{P}^+ = \mathbb{P} \cup \{B\}$. If $\text{cls}(B) = 0$ then $\{B\}$ is a d-basic set of $\mathbb{P}^+$ and has rank lower than that of $\mathbb{B}$. Suppose otherwise $\text{cls}(B) = p > 0$. As B is reduced w.r.t. $\mathbb{B}$, then there exists an $i(1 \le i \le r)$ such that $p \le \text{cls}(B_i)$ and $p > cls(B_{i-1})$ when $i > 1$. Moreover, in the case $p = cls(B_i)$, we have $\deg(B, x_p) < \text{lead}(B_i)$. Hence $\{B_1, B_2, \dots, B_{i-1}, B_i\}$ is an d-ascending set contained in $\mathbb{P}^+$ and has rank lower than that of $\mathbb{B}$. The d-basic set of $\mathbb{P}^+$has therefore rank lower than that of $\mathbb{B}$.

Definition 2.1: A finite non-empty ordered set $\mathbb{A} = \{A_1, A_2, \dots, A_r\}$ of non-zero d-polynomial in $\mathcal{K}\{x\}$ is called a d-ascending if we have either $r = 1$ or $r > 1$ and $0 < \text{cls}(A_1) < \text{cls}(A_2) < \dots < \text{cls}(A_r)$.

Definition 2.2: Let $\mathbb{A}$ be as above and call

d-prem$(Q, \mathbb{A})$ =d-prem (...d-prem$(Q, A_r), \dots, A_1)$

the differential pseudo-remainder (d-pseudo-remainder) of Q w.r.t $\mathbb{A}$. For any d-polynomial set $\mathbb{P}$ we define d-prem$(\mathbb{P}, \mathbb{A}) = \{d - prem(P, \mathbb{A}) | P \in \mathbb{P}\}$, $\text{ini}(\mathbb{P}) = \{\text{ini}(P) | P \in \mathbb{P}\}$ and $\text{sep}(\mathbb{P}) = \{\text{sep}(P) | P \in \mathbb{P}\}$.

Definition 2.3: A d-ascending set $\mathbb{C}$ is called a d-char set of a d-polynomial set $\mathbb{P}$ if $\mathbb{C}$ is contained in the d-ideal generated by the d-polynomials in $\mathbb{P}$ and d-prem$(\mathbb{P}, \mathbb{C}) = 0$.

From the definition it is easy to establish the following zero relation: Zero$(\mathbb{P})$ = Zero $(\mathbb{C}/\mathbb{I}) \cup \bigcup_{I \in \mathbb{I}}$ Zero $(\mathbb{P} \cup I)$, where $\mathbb{I} = \text{ini}(\mathbb{C}) \cup \text{sep}(\mathbb{C})$.

3. ADMISSIBLE DIFFERENTIAL REDUCTIONS ANDGENERALIZED DIFFERENTIAL CHARACTERISTIC SETS

A. Generalized differential characteristic sets

In this section we define the generalized d-char sets for the ordinary d-polynomial sets. The d-char set method has several variants. We use notion of the differential medial set (d-medial set) of the d-polynomial set $\mathbb{P}$.

*Definition 3.1:*Let $\mathbb{P}$ be a non-empty d-polynomial set $\mathcal{K}\{x\}$. A d- ascending set $\mathbb{M}$ is called a d-medial set of $\mathbb{P}$, if $\mathbb{M} \subset \langle \mathbb{P} \rangle$and $\mathbb{M}$ has ranking not higher than the ranking of any d-basicset of $\mathbb{P}$.

The d-medial sets are the d-ascending sets with rank not higher than that of the d-basic set of $\mathbb{P}$ and in which all the d- polynomials are linear combinations of the d-polynomials in $\mathbb{P}$ with d-polynomial coefficients. Therefore any d-basic set itself is a special d-medial set of the d-polynomial set. It has been proved that in Ritt's original algorithm, the d-basic set can be replaced by the d-medial set.

Proposition 3.1: Let P be a non-empty d-polynomial set in $\mathcal{K}\{x\}$, $\mathbb{M} = \{M_1, M_2, \dots, M_r,\}$ be a d-medical set where $\text{cls}(M_1) > 0$, If M is a non-zero d-polynomial reduced w.r.t. $\mathbb{M}$. Then any d-medial set $\mathbb{M}^+$ of the d-polynomial set $\mathbb{P}^+ = \mathbb{P} \cup \mathbb{M} \cup \{M\}$ has rank lower than that of $\mathbb{M}$ i.e.,$\mathbb{M}^+ \prec \mathbb{M}$.

Proof: Let $\mathbb{B}^+$ and $\mathbb{B}^*$ be d-basic sets of $\mathbb{P}^+$and $\mathbb{P} \cup \mathbb{M}$ respectively. Then $\mathbb{B}^* \preccurlyeq \mathbb{M}$ by definition. If $\mathbb{B}^* \sim \mathbb{M}$, then M is reduced w.r.t. $\mathbb{B}^*$. Hence by Definition 3.1 and Lemma 2.1 we have $\mathbb{M}^+ \preccurlyeq \mathbb{B}^+ \preccurlyeq \mathbb{B}^* \sim \mathbb{M}$>. If $\mathbb{B}^* \prec \mathbb{M}$, then $\mathbb{M}^+ \preccurlyeq \mathbb{B}^+ \preccurlyeq \mathbb{B}^* \prec \mathbb{M}$ hold. Therefore in either case $\mathbb{M}^+ \prec \mathbb{M}$. Which completes the proof.

There may appear some redundant factors which are the initials of other occurring d-polynomials during computation of the d-char sets, these factors must be removed for controlling the expansion of d-polynomial size. If the removal of such factors is allowed then the d-ascending sets which are computed by the d-char set algorithm are no longer Ritt's d-char sets, instead they are what we call the modified d-char sets.

Definition 3.2: For any non-empty d-polynomial set $\mathbb{P}$ in $\mathcal{K}\{x\}$, an ascending set $\mathbb{C}$ is called a generalized differential characteristic (generalized d-char) set if it satisfied the following twoconditions

(1) $\mathbb{C} \subset \langle \mathbb{P} \rangle$
(2) There exists a d-polynomial set$\mathbb{Q} \subseteq \mathcal{K}\{x\}$ such that$\langle \mathbb{Q} \rangle = \langle \mathbb{P} \rangle$ and d-prem $(\mathbb{Q}, \mathbb{C}) = 0$

In our paper, the generalized d-char sets for the ordinary d-polynomial sets are defined in order to preserve the ideal relations by means of the admissible *differential reductions* (d-reductions). When only the zeros of d-polynomial sets are of concern, we consider the radical ideal relation instead to define and compute the generalized d-char sets.

B. Admissible d-reductions

The standard d-reduction which is used in Ritt-Wu's algorithm for the d-char sets and its variants is the d-pseudo division, that may lead to the superfluous factors and swell of the d-polynomial coefficients. This is among the main reasons that degrades the performance of Ritt-Wu's algorithm in many cases. We introduce some specific admissible d-reductions and discuss implementation of our scheme by means of the admissible d-reductions. We define some functions before introducing the admissible d-reductions.

Definition 3.3: Let P and Q be two non-zero d-polynomials in $\mathcal{K}\{x\}$ and $<_{lex}$ be a lexicographic term order in $\mathcal{K}\{x\}$ such that $x_1 < x_2 < \dots < x_n$. We order $P < Q$ or $Q < P$ if

(1) $\text{lead}(P) <_{lex} \text{lead}(Q)$, or
(2) $\text{lead}(P) = \text{lead}(Q)$ and $\text{rank}(P, \text{lead}(P)) < \text{rank}(Q, \text{lead}(Q))$

If neither $P < Q$ nor $P > Q$, then we write $P \approx Q$. In addition, $P \le Q$ is defined.

Note that the partial order $<$ defined above is refinement of $\prec$ defined in Definition 2.1. In particular, for any two d-polynomials $P, Q \in \mathcal{K}\{x\} \backslash \{0\}$, $P \prec Q$ implies that $P < Q$ on the contrary, $P < Q$ implies only $P \precsim Q$, and $P \prec Q$ does not essential hold.

Definition 3.4: Let $\mathfrak{D}$ be defined, as an operation between two non-zero d-polynomials $P, Q \in \mathcal{K}\{x\} \backslash \mathcal{K}$. It takes d-polynomials as input and returns an ordered set of two d-polynomials as output. This output is denoted by d-Prem $(P, Q, \mathfrak{D})$.

Definition 3.5: Let us define a function d-Rem as $[R_1, R_2] =$ d-Rem $(P, Q, \mathfrak{D})$ for d-polynomials $P, Q \in \mathcal{K}\{x\} \backslash \mathcal{K}$. The operation $\mathfrak{D}$ is called an *admissible d-reduction* in $\mathcal{K}\{x\}$ in $\mathcal{K}\{x\}$ if $R_1, R_2 \in \langle P, Q \rangle$.

Suppose the $\mathfrak{D}$ is an admissible d-reduction in $\mathcal{K}\{x\}$, we see that P is $\mathfrak{D}$-reducible w.r.t. Q if $P > R_1$ and $Q \ge R_2$;

otherwise Q is said to be $\mathfrak{D}$-reducible w.r.t. P. The d-polynomials P and Q are called *d-reductend* and the *d-reductor* respectively. The function d-Rem $(P,Q,\mathfrak{D})$ is called the $\mathfrak{D}$ *d-reduction rest* of P by Q.

Definition 3.6: Let *d-RemCh* be an extension of function d-Rem which is defined earlier. D-RemCh$(P,Q,\mathfrak{D})$ not only returns the $\mathfrak{D}$ d-reduction rest of P by Q, but it also returns a Boolean value b. When this b is true, the d-reduction rest $\mathfrak{R}$ must satisfy the condition $P,Q \in \langle\mathfrak{R}\rangle$.

If $\mathfrak{D}$ is an admissible reduction in $\mathcal{K}\{x\}$, then for any two d-polynomials P and Q, we say that P is $\mathfrak{D}$-reducible w.r.t. Q if $P > R_1$ and $Q \geq R_2$. Otherwise P is d–reduced with respect to Q. The P and Q are called the *d-reductend* and the *d-reductor* respectively. d-Rem $(P,Q,\mathfrak{D})$ is called the d-reduction rest of d-polynomial P by d-polynomial Q.

Definition 3.7: dFind3R $(\mathbb{P},\mathbb{D})$ is defined as a function which chooses d-polynomial P (d-reductend) and Q (d-reductor) from $\mathbb{P}$ and $\mathfrak{D}$ (d-reduction) from $\mathbb{D}$ such that d-polynomial P is $\mathfrak{D}$-reducible with respect to d-polynomial Q. Moreover if d-Find3 $(\mathbb{P},\mathbb{D})$ finds appropriate $P,Q,\mathfrak{D}$ then it returns this triple otherwise it will return [ ].

Definition 3.8: d-Rem$(P,Q,\mathfrak{D})$ is a function which returns $\mathfrak{D}$ (reduction-rest) $\mathfrak{R}$ of d-polynomial P by the d-polynomial Q and a Boolean value b. Whenever this b is true, the reduction –rest $\mathfrak{R}$ must satisfy the condition $P,Q \in \langle\mathfrak{R}\rangle$.

Now we define the admissible reductions we have used in our algorithm.

Definition 3.9: **Univariate GCD d-reduction**.

Univariate GCD d-reduction is defined as follows

d-Rem$(P,Q,\mathfrak{D}_{UG}) \coloneqq [0, \gcd(P,Q,x_q)]$ If P,Q are univariate polynomials in x_q .

d-Rem$(P,Q,\mathfrak{D}_{UG}) \coloneqq [P,Q]$, otherwise x_q is some variable in x.

It can be easily verified from the definition of gcd that $\mathfrak{D}_{UG}$ is an admissible d-reduction and d-polynomial P is $\mathfrak{D}_{UG}$-reducible w.r.t. d-polynomial Q if and only if P and Q are univariate d-polynomial in the same variable.

Jacobi introduced pseudo-division by multiplied a polynomial f with a certain power of the leading coefficient of g before performing the division with remainder. So using pseudo-division instead of division with remainder in every step in the Euclidean Algorithm yields an algorithm with all intermediate results. Pseudo-division which is used mostly in many triangular decomposition algorithms is an admissible reduction also used in our algorithm for differential polynomial systems.

Definition 3.10: **d-pseudo-division reduction.**

The d-pseudo-division reduction is defined below.

$$\text{d-Rem}(P,Q,\mathfrak{D}_{PD}) \coloneqq [d-prem(P,Q,lead(Q)),Q]$$

Then we have

$$\operatorname{sep}(Q)^{\alpha}\operatorname{ini}(Q)^{\beta}P = \operatorname{pquo}(P,Q,\operatorname{lead}(Q))Q + \operatorname{prem}(P,Q,\operatorname{lead}(Q))$$

where α,β are non-negative integers. Therefore

$$d-Rem(P,Q,\mathfrak{D}_{PD}) \subseteq \langle P,Q\rangle,$$

which ensures that $\mathfrak{D}_{PD}$ is an admissible reduction in $\mathcal{K}\{x\}$.

There is one big disadvantage of the d-pseudo-division is that it leads to exponential coefficient growth; the coefficient of the intermediate results are generally enormous, their bit length may possibly be exponential in the bit length of the input d-polynomials f and g. Thus we also use the one stepd-pseudo-division as a reduction for d-polynomials.

Definition 3.11: Let $P$and Qbe two d-polynomials in $\mathcal{K}\{x\}\backslash\mathcal{K}$ with $.x_q = \operatorname{lead}(Q)$, $I = \operatorname{ini}(Q)\operatorname{sep}(Q)$ and $J = \operatorname{lc}(P,x_q)$. Suppose that P is d-reducible with respect to Q. Then one-step d-pseudo-division can be performed as following

$$R \coloneqq FP - GQ(x_q)^{\deg(P,x_q)-\operatorname{ld}(Q)}$$

where $F = \operatorname{lcm}(I,J)/J$ and $G = \operatorname{lcm}(I,J)/I$, R is called the one-step d-pseudo-remainder of d-polynomial P w.r.t. d-polynomial Q. It is denoted by d-stprem(P,Q).

It can be seen that the d-pseudo-division is a recursive application of the one-step d-pseudo-division. Therefore it may immediately lead to superfluous factors of the d-pseudo-remainders but for the one-step d-pseudo-division it is easy to control the size of d-polynomials, which may output as smaller d-reduction-rests than the d-pseudo-division. One may use one-step d-pseudo-division reduction as admissible reduction.

Definition 3.12: **One-step d-pseudo-division reduction**.

One-step d-pseudo-division reduction is defined as follows:

d-Rem$(P,Q,\mathfrak{D}_{SP}) \coloneqq [d-stprem(P,Q),Q]$ if P be d-reducible w.r.t Q.

d-Rem$(P,Q,\mathfrak{D}_{SP}) \coloneqq [P,Q]$ otherwise.

Let P be d-reducible w.r.t Q then d-stprem $(P,Q) \in \langle P,Q\rangle$. Therefore

d-Rem$(P,Q,\mathfrak{D}_{SP}) \subseteq \langle P,Q\rangle$. Otherwise

d-Rem$(P,Q,\mathfrak{D}_{SP}) = [P,Q] \subseteq \langle P,Q\rangle$ is obvious. Hence $\mathfrak{D}_{SP}$ is an admissible reduction by definition.

Definition 3.13: **One-step d-division reduction.**

The division operation can also be viewed as an admissible reduction. So we define the one-step d-division reduction.

d-Rem$(P,Q,\mathfrak{D}_{SD}) \coloneqq [P - \frac{M}{\operatorname{lead}(Q)} \cdot Q, Q]$ if $\exists$ a monomial M of $P$and if $\exists$ a monomial M of P such that $\operatorname{lead}(Q)|M$.

d-Rem$(P,Q,\mathfrak{D}_{SD}) \coloneqq [P - \frac{M}{\operatorname{lead}(Q')} \cdot Q', Q]$ if $\exists$ a monomial M of P such that $\operatorname{lead}(Q')|M$.

d-Rem$(P,Q,\mathfrak{D}_{SD}) \coloneqq [P,Q]$, otherwise.

It can be verified that $\mathfrak{D}_{SD}$is an admissible reduction and d-polynomial $P$is $\mathfrak{D}_{SD}$-reducible w.r.t d-polynomial Qif andonly if there exists a monomial of Pwhich can be divided bylead(Q) or lead(Q').

Subresultants and polynomial remainder sequences are an important tool in the field of polynomial computer algebra. Multi polynomial resultants are the oldest known methodology for eliminating the variables. Computation techniques to manipulate the sets of polynomial equations are gaining importance in the symbolic and the numeric computations.

The fundamental difficulties include simultaneous elimination of one or more variables to acquire a symbolically smaller system and figuring the numeric solutions of the system of equations. The method of resultant is one of such methods. The main consequence is the construction of a single resultant polynomial of n homogenous equations in n unknowns. We refer this resultant as the multi polynomials resultant of certain system of equations.

Let two d-polynomials P and Q in $\mathcal{K}\{x\}$ are said to be *similar* if there exist $a, b \in \mathcal{K}$, $ab \neq 0$, such that $aP = bQ$, denoted by $P \sim Q$.

Definition 3.13: Let the polynomial P and Q be renamed P_1 and P_2 and suppose that $\deg(P_1, x_p) \geq \deg(P_1, x_p)$. Then we form a sequence of d-polynomials P_1, P_2, …, P_r such that $P_i \sim$d-prem(P_{i-2}, P_{i-1}, x_p), $i = 3, \ldots, r$ and d-prem$(P_{r-1}, P_r, x_p) = 0$. Such a sequence is called a d-polynomial remainder sequence of P and Q w.r.t. x_p.

Let $\text{lvar}(P) = \text{lvar}(Q)$, $\text{ld}(P) \geq \text{ld}(Q)$ and treat P, Q univariate d-polynomial in $x_q = \text{lvar}(Q)$ with coefficients in $\mathcal{K}\{x_1, x_2, \ldots, x_{q-1}\}$. Let P and Q be d-polynomials, define d-res(P, Q, x_q) be the resultant of P and Q w.r.t.x_q. Then if we have d-res$(P, Q, x_q) = 0$, it means P_r is the greatest common divisor of P and Q w.r.t. x_q. Also, in the case when d-res$(P, Q, x_q) \neq 0$, we have cls$P_r <$ cls(P) and cls$(P_{r-1}) =$ cls(P). Thus P_{r-1} is the greatest common divisor of P and Q with respect to x_q. Note that we only mention definitions required. For details of the resultants and subresultants reader may see [2] and [21].

Definition 3.15: **Subresultant d-PRS reduction.**
We introduce the following d-reduction for d-polynomials.
d-Rem$(P, Q, \mathfrak{D}_{PD}) = [0, P_r]$ if $\text{lead}(P) = \text{lead}(Q), \text{ld}(P) = \text{ld}(Q)$, and d-res $(P, Q,$lead $(Q)) = 0$.
d-Rem$(P, Q, \mathfrak{D}_{PD}) = [P_r, P_{r-1}]$ if $\text{lead}(P) = \text{lead}(Q), \text{ld}(P) = \text{ld}(Q)$, and d-res $(P, Q,$lead $(Q)) \neq 0$.
d-Rem$(P, Q, \mathfrak{D}_{PD}) = [P, Q]$ otherwise.

After defining the admissible d-reductions above we are ina position to describe an algorithmic scheme for computing generalized d-char sets for ordinary d-polynomial sets.

The scheme was purposed by Jin et all [17] earlier for algebraic case. Our purpose is to find a concrete algorithm to triangularize d-polynomial sets with the help of admissible d-reductions and some powerful elimination strategies in order to replace the algorithm of d-basic set which is used in Ritt-Wu algorithm.

4. COMPUTING GENERALIZED ORDINARY DIFFERENTIALCHARACTERISTIC SETS

A. Algorithmic scheme

Jin *et al* [17] weakened the compulsory condition by replacing the given polynomial set with an arbitrary polynomial set which generates the same ideal as the input polynomial set, leading to the concept of generalized characteristic set. This set may have polynomials of degrees smaller than the degrees of those in the input set and consequently may take less computing time for pseudo-reduction to 0. They made use of this weakening condition and followed the work [6] and presented an algorithmic scheme for computation of generalized characteristic sets by means of admissible reductions other than pseudo-division, for algebraic case only. This scheme is valuable as it effectively control the swell of polynomial coefficients as well as degrees. Therefore, we use the same idea for ordinary d-polynomial sets. For this purpose, admissible d-reductions other than pseudo-division are used in order to control the swell of coefficients of intermediate d-polynomials during the computation of d-char sets. In order to compute generalized d-char sets, we propose an algorithmic scheme which has set of admissible d-reductions, d-medial sets and strategies for finding the d-reduction polynomials as placeholders. Our first algorithm NewdCharSet1 computes a generalized d-char set for any given ordinary d-polynomial set.

Algorithm 1: NewdCharSet for computing a generalized d-char set of any given d-polynomial set.
Input:$\mathbb{P}$ = a non-empty d-polynomial set in $\mathcal{K}\{x\}$; $\mathbb{D}$= set of admissible reductions in $\{x\}$.
Output:$\mathbb{C}$ = generalized d-char set of *PS*.
$\mathbb{Q} \coloneqq \mathbb{P}$
$\mathbb{R} \coloneqq \mathbb{P}$
while $\mathbb{R} \neq 0$ do
    $[\mathbb{C}, B] \coloneqq dMedSet(\mathbb{Q}, \mathbb{D})$;
if $\mathbb{C}$ is *contradictory* **then**
    $\mathbb{R} \coloneqq 0$
else
$\mathbb{R} \coloneqq$d-rem $(B \backslash \mathbb{C}, \mathbb{C} \backslash \{0\}$;
$\mathbb{Q} \coloneqq \mathbb{Q} \cup \mathbb{C} \cup \mathbb{R}$;

Proof. (NewdCharSet) *Correctness*.The correctness of algorithm can be seen as following. We know from the properties of dMedSet's output that $\langle B \rangle = \langle \mathbb{Q} \rangle$ and $\mathbb{C}$ is d-medial set of $\mathbb{Q}$. It means $\mathbb{C} \subseteq \mathbb{Q}$. Now according to d-pseudo division formula we have $\mathbb{R} \subseteq \langle \mathbb{Q} \rangle$ always holds during running of algorithm. Also ideal generated by $\mathbb{Q}$ remains the same with ensure that $\langle \mathbb{Q} \rangle = \langle \mathbb{P} \rangle$ always true. In addition d-prem$(\mathbb{Q}, \mathbb{C}) = \{0\}$ when the ***while*** loop terminate. Hence $\mathbb{C}$ is the d-generalized char set of $\mathbb{P}$ by definition.
Termination. We use $\mathbb{C}_i$ and $\mathbb{Q}_i$ to represent the values in the *ith* ***while*** loop for $\mathbb{C}$ and $\mathbb{Q}$ respectively. If we recall the properties of d-medical set, we obtain a sequence of d-triangular set $\mathbb{M}_1 \prec \mathbb{M}_2 \prec \mathbb{M}_3 \prec$. Also we know the sequence of d-triangular set is finite. Hence the algorithm terminates.

B. Algorithm for computing d-medial sets

Here algorithm dMedSet is presented which replaces the d-basic set algorithm of Wu d-char sets. It uses several admissible d-reduction instead of apply simple d-pseudo-division. Therefore, the computation process may speed up and also we may get simple output for larger problems.

Algorithm 2:dMedSet for computing a d-medial (d-basic) set of any given d-polynomial set
Input: $\mathbb{P}$ = any given polynomial set in $\mathcal{K}\{x\}$; $\mathbb{D}$ = setof admissible reductions.
Output: $\mathbb{M}$ = d-medial set of $\mathbb{P}$.

$\mathbb{Q} \coloneqq \mathbb{P}$;
$\mathbb{R} \coloneqq \mathbb{P}$;
while *cond* **do**
  $[P, Q, \mathfrak{D}] \coloneqq$ dFind3R$(\mathbb{Q}, \mathbb{D})$
  $[\mathfrak{R}, b] \coloneqq$ d-RemCh$[P, Q, \mathfrak{D}]$;
  if$\mathfrak{R} \in \mathcal{K}\backslash\{0\}$ then
  $\mathbb{Q} \coloneqq \{1\}$; $\mathbb{R} \coloneqq \{1\}$;
  break;
else
  $\mathbb{Q} \coloneqq \mathbb{Q} \backslash\{P, Q\} \cup \mathfrak{R}$;
  If $P, Q \in \mathbb{R}$ and b **then**
  $\mathbb{R} \coloneqq \mathbb{R} \cup \{P, Q\} \cup \mathfrak{R}$;
$\mathbb{M} \coloneqq$dBasSet $(\mathbb{Q} \cup \mathbb{P})$

Proof. (dMedSet) *Correctness*. This is clear from the statement$[\mathfrak{R}, b] \coloneqq$ d-RemCh $(P, Q, \mathfrak{D})$ that $RS \subset \langle P, Q\rangle$ by Definition 3.6. It is easy to verify that $\mathbb{Q} \subseteq \langle\mathbb{P}\rangle$ will always hold during the running of the algorithm, therefore $\mathbb{M} \subseteq \langle\mathbb{P}\rangle$. In addition, the ranking of $\mathbb{M}$ is not higher than that of any d-basic set of $\mathbb{P}$; therefore $\mathbb{M}$ is a d-medial set of $\mathbb{P}$ by definition. Let $\mathbb{R}_i$ be the initial value of $\mathbb{R}$ in the *ith* **while** loop. Then obviously, $\langle\mathbb{R}_1\rangle = \langle\mathbb{P}\rangle$. Assume that $\langle\mathbb{R}_i\rangle = \langle\mathbb{P}\rangle$ and we claim that $\langle\mathbb{R}_{i+1}\rangle = \langle\mathbb{P}\rangle$ as follows.

Let the statement $[\mathfrak{R}, b] \coloneqq$ d-Rem$(P, Q, \mathfrak{D})$, in the *ith* **while** loop. Suppose we have the Boolean expression $(P, Q \in RS$ and $b)$ equals **false**, then$\mathbb{R}_{i+1} = \mathbb{R}_{i.}$ Hence $\langle\mathbb{R}_{i+1}\rangle = \langle\mathbb{P}\rangle$. Furthermore, assume that $(P, Q \in \mathbb{R}$ and $b)$ equals **true**, then we have $\mathbb{R}_{i+1} = \mathbb{R}_i\backslash\{P, Q\} \cup \mathfrak{R}$. Since $\mathfrak{D}$ is an admissible d-reduction, therefore $\mathfrak{R} \subseteq \langle P, Q\rangle$. It is known by the assumption on b returned by dRemCh, one has $P, Q \in \langle\mathfrak{R}\rangle$. Thus $\langle P, Q\rangle = \langle\mathfrak{R}\rangle$, which implies that $\langle\mathbb{R}_{i+1}\rangle = \langle\mathbb{R}_i\backslash\{P, Q\} \cup \mathfrak{R}\rangle$. Moreover, from the above $\langle\mathbb{R}_{i+1}\rangle = \langle\mathbb{P}\rangle$, therefore $\langle\mathbb{R}\rangle = \langle\mathbb{P}\rangle$ always holds.

Hence the correctness of the algorithm is proved.

Termination. Termination of algorithm is obvious by the assumption on cond.

In the algorithm dMedset *cond*, $\mathfrak{D}$ and dFind3R are Procurators. There are many ways to assign cond. In case when weset it cond = false the while loop in the algorithm does notstart and this algorithm becomes identical to the d-basic set algorithm of Ritt-Wu. By taking different d-medial sets we can have different variants of d-char set algorithm. In particular, if we replace the d-medial set by the d-basic set then the generalized d-char set algorithm is identical to the d-char set algorithm. In addition the dMedSet not only gives us a d-medial set but also produces another d-basis B of the ideal$\langle\mathbb{Q}\rangle$.

C. Algorithm for finding triples of d-reductends, d-reductors and d-reductions

In this section we present the algorithm dFind3R for finding triples of d-reductends, d-reductors and d-reductions. Here we discuss the possible strategies for dFind3R. In our execution, we have cond to be dFind3R $(\mathbb{Q}, \mathbb{D}) \neq \phi$ therefore while looprepeats till we get triple $[P, Q, \mathfrak{D}]$ such that d-polynomial $P$is $\mathfrak{D}$-reducible w.r.t. d-polynomial Q.

We define dFind3R, as we have described above the set of selected admissible d-reduction is $\mathbb{D} = \{\mathfrak{D}_{UG}, \mathfrak{D}_{SD}, \mathfrak{D}_{SC}, \mathfrak{D}_{SP}\}$. Our purpose here is to select thebest triple as there may exist many triples between a pair of d-polynomials. So we define here a way to order the triples. We take the d-polynomial of maximal class and maximal degrees first reduced by a d-polynomial of minimal degree.

At first, it is usual that a triple with $\mathfrak{D}_{UG}$ is better thanthe triples without having it. Secondly, a triple with$\mathfrak{D}_{SD}$is better than the remaining two as it is easy to see thatcomputation of the$\mathfrak{D}_{SD}$ reduction rest of the d-polynomial P and Q needs less multiplications and reduction rest is mostly smaller. Thirdly, the triple with the $\mathfrak{D}_{SC}$ is may be better thantriple with $\mathfrak{D}_{SP}$ as $\mathfrak{D}_{SC}$ involves coefficients of small in sizethan that of $\mathfrak{D}_{SP}$. Finally, a triple $[P, Q, \mathfrak{D}]$may be betterthan the triple $[P', Q', \mathfrak{D}]$ $(\mathfrak{D} \in \mathbb{D})$ if $P > P'$ or $Q > Q'$.This observation base on the method of d-reduction in Ritt-Wu algorithm. That is the d-polynomial of maximum class and maximum degree should be first reduced by the d-polynomial having minimal degree. In short we provide the formal definition of an order on the triples: $[P, Q, \mathfrak{D}] < [P', Q', \mathfrak{D}']$ if $\mathfrak{D} < D'$ or $\mathfrak{D} = \mathfrak{D}'$and $P < P'$ or $\mathfrak{D} = \mathfrak{D}'$ and $P = P'$and $Q = Q'$. We order the admissible d-reductions in $\mathbb{D}$ as$\mathfrak{D}_{UG} > \mathfrak{D}_{SD} > \mathfrak{D}_{SC} > \mathfrak{D}_{SP}$. By taking a d-polynomial set $\mathbb{P}$ and a set $\mathbb{D}$of admissible d-reductions as an input inalgorithm dFind3R, we take the following steps to find a best triple if exists or $\emptyset$as output:

Algorithm 3: dFind3R for computing a minimal triple$[P, Q, \mathfrak{D}_{SD}]$,such that P is $\mathfrak{D}$-reducible w.r.t Q
Input: A set of d-polynomial in $\mathcal{K}\{x\}$;
$$\mathbb{D} = \{\mathfrak{D}_{UG}, \mathfrak{D}_{SD}, \mathfrak{D}_{SC}, \mathfrak{D}_{SP}\}$$
Output: $\emptyset$ or $[P, Q, \mathfrak{D}_{SD}]$ such that P is $\mathfrak{D}$-reducible w.r.t Q.
$S \coloneqq \emptyset$;
for$i = n$ to 1;
while$|S| < 2$ do
  $S \coloneqq \{P \in \mathbb{P}: P \in \mathcal{K}\{x_I\}\}$;
if$\exists P, Q \in SP \neq Q$ such that P is $\mathfrak{D}_{UG}$-reducible w.r.t. Q **then**
  return $[P, Q, \mathfrak{D}_{SD}]$ such that $P \in S$ has maximal degree and $Q \in S\backslash\{P\}$ has fewest terms and minimal degree, P is $\mathfrak{D}_{UG}$-dreducible w.r.t Q;
$[P_1, P_2, \ldots, P_r] \coloneqq \mathbb{P}$
for $i = r$ to 2 **do**
  $Q \coloneqq \{Q \in \mathbb{P}\backslash\{P_i\}: P_i \text{is} \mathfrak{D}_{SD} - \text{dreducible w.r.t } Q\}$;
  if$Q \neq \emptyset$ **then**
    choose $Q \in Q$ fewest terms and minimal leading degree;
    return $[P_i, Q, \mathfrak{D}_{SD}]$;
for $i = r$ to 2 **do**
  $Q \coloneqq \{Q \in \mathbb{P}\backslash\{P_i\}: P_i \text{is} \mathfrak{D}_{SP} - \text{dreducible w.r.t } Q\}$;
  if$Q \neq \emptyset$ **then**
    choose $Q \in Q$ fewest terms and minimal leading degree;
    return $[P_i, Q, \mathfrak{D}_{SP}]$;
for $i = r$ to 2 **do**
  $Q \coloneqq \{Q \in \mathbb{P}\backslash\{P_i\}: P_i \text{is} \mathfrak{D}_{PD} -$
dreducible w.r.t $Q\}$;
  If $Q \neq \emptyset$ **then**
    choose $Q \in Q$ fewest terms and minimal leading degree;
    return $[P_i, Q, \mathfrak{D}_{SC}]$;

∅

Step 1. At first it verifies if there exist triples $[P, Q, \mathfrak{D}_{UG}]$,where the d-polynomials $P, Q \in \mathbb{P}$, $P \neq Q$. If there is onlyone triple then it will return it, in case there are several, thenit will return the triple in which d-polynomial P has maximalclass and maximal leading degree and d-polynomial Q hasfewest number of terms and of minimal degree.

Step 2. In the second step the d-polynomials in $\mathbb{P}$ will besorted increasingly w.r.t. partial order < and then it will taked-polynomial P of highest order, and check if there existsome d-polynomial Q such that P is $\mathfrak{D}_{SD}$ d-reducible w.r.td-polynomial Q. Now if only one Q then return $[P, Q, \mathfrak{D}_{SD}]$,else return Q which has minimum terms and least leadingdegree; otherwise in case when there is no such triple exist,go to next step.

Step 3. Again find d-polynomial P of highest order, thencheck if there exist some d-polynomial Q such that P is$\mathfrak{D}_{SP}$ d-reducible w.r.t d-polynomial Q. If exist triple return$[P, Q, \mathfrak{D}_{SC}]$, else go to next step following same procedureas step 2.

Step 4. Similarly, start with a d-polynomial P of highest orderagain, then check if there exists any d-polynomial Q such that$P$ is $\mathfrak{D}_{SP}$ d-reducible w.r.t d-polynomial Q. If exist triplereturn $[P, Q, \mathfrak{D}_{PD}]$, else go to next step.

Step 5. If there does not exist some triple in all four steps,then output will be ∅.

It is significant to note that the ordering is critical for the efficiency of the algorithm dMedSet. One may sort d-polynomials w.r.t. different orderings depending on the admissible d-reductions. Thus by changing the ordering, we can get different d-char sets for a d-polynomial set.

The above steps can be described formally in the following algorithm dFind3R which finds the triples of d-reductends, d-reductors and d-reductions for a d-polynomial set.

Proof.(dFind3R) Correctness and termination of the dFind3Ralgorithm follows from the analysis and description described above in steps.

The design of this dFind3R is quite flexible and it can be made more technical and widespread. If the input set of the algorithm dFind3R contains d-polynomials of some particular form or configuration, then new d-reductions approaches maybe implemented in order to improve the efficiency. One might introduce other admissible d-reductions for example; one may use modular d-reduction. There are numerous ways to improve this algorithm. Other techniques can be smeared, for example from linear algebra.

5. IMPLEMENTATION AND EXPERIMENTS

In this section we present some examples to compute the generalized d-char sets of ordinary d-polynomial sets by implementing our algorithm NewdCharSet described in Section4 using the epsilon library of Wang in Maple 14.

Example 5.1: [4,6,18,23]. Kepler-Newton's Laws. Let the coordinate of the planet be (x, y), depending on the time variable t. Assume that the sun is located at the origin(0, 0). Then the d-polynomial equations for the Newton's law are

$$N_1 = (r^2 a)' = 0, N_2 = xy'' - x''y = 0$$

where a is the acceleration of the planet and r the length ofthe radius vector from the sun to the planet. Then we have

$$K_1 = r^2 - x^2 - y^2 = 0, \;\; K_2 = a^2 - x''^2 - y''^2 = 0$$

Take a d-polynomial set $\mathbb{N} = \{N_1, N_2, K_1, K_2\}$ with $x \prec y \prec r \prec a$. If we compute a d-char set of $\mathbb{N}$, by the NewdCharSet algorithm, we get the following output:

$\{-324 x^9x''^4xy^2x'''x''''-324 x^7x''4x'^2y^4x'''x''''-$
$108 x^5x''^4x'^2y^6x'''x''''-486 x'''^2x^{10}x'^3y^2x'x''''-$
$486x'''^2x^8x'^3y^4x'x''''-162 x'''^2x^6x'^3y^6x' x''''+$
$108 x^{10}x''^4x'y^2x'''x'''''+ 108 x^8x''^4x'y^4x'''x'''''+$
$36 x^6x''^4x'y^6x'''x'''''+ 216 x^7x''^7x'^3y^2-48 x^6x''^6x'^5y^2 +$
$216 x^5x''^7x'^3y^4-48 x^4x''^6x'^5y^4 + 72 x^3x''^7x'^3y^6-$
$16 x^2x''^6x'^5y^6+216 x^8x''^6x'^2y^2x''' + 108 x^9x''^5x'^2y^2x'''''+$
$102 x^9x''^3x'^2y^2x'''^3-120 x^7x''^5x'^4y^2x'''+$
$216 x^8x''^5x'^3y^2x''''-444 x^8x''^4x'^3y^2x'''^2 +$
$216 x^6x''^6x'^2y^4x'''+ 108 x^7x''^5x'^2y^4x'''''+$
$102 x^7x''^3x'^2y^4x'''^3-120 x^5x''^5x'^4y^4x'''+$
$216 x^6x''^5x'^3y^4x''''-444 x^6x''^4x'^3y^4x'''^2 +$
$72 x^4x''^6x'^2y^6x'''+ 36 x^5x''^5x'^2y^6x'''''+$
$34 x^5x''^3x'^2y^6x'''^3-40 x^3x''^5x'^4y^6x'''+$
$72 x^4x''^5x'^3y^6x''''-148 x^4x''^4x'^3y^6x'''^2 +$
$36 x'''x^{12}x''^4x'x'''''-108 x'''x^{11}x''^4x'^2x''''-$
$162 x'''^2x^{12}x''^3x'x''''+ 54 x'''^2x^9x''^5y^2x'+$
$27 x'''^2x^{11}x''^3y^2x'''''+ 390 x'''^4x^{10}x''^2y^2x'-$
$135 x'''^3x^{11}x''^2y^2x''''+ 54 x'''^2x^7x''^5y^4x'+$
$27 x'''^2x^9x''^3y^4x'''''+ 390 x'''^4x^8x''^2y^4x'-$
$135 x'''^3x^9x''^2y^4x''''+ 18 x'''^2x^5x''^5y^6x'+$
$9 x'''^2x^7x''^3y^6x'''''+ 130 x'''^4x^6x''^2y^6x'-$
$45 x'''^3x^7x''^2y^6x''''+ 40 x'''^5x^{13}x''+ 72 x^9x''^7x'^3-$
$16 x^8x''^6x'^5 + 18 x'''^2x^{11}x''^5x'+ 72 x'''x^{10}x''^6x'^2 +$
$130 x'''^4x^{12}x''^2x'-148 x'''^2x^{10}x''^4x'^3+34 x'''^3x^{11}x''^3x'^2-$
$40 x'''x^9x''^5x'^4 + 120 x'''^5x^{11}x''y^2 + 120 x'''^5x^9x''y^4 +$
$40 x'''^5x^7x''y^6 + 9 x'''^2x^{13}x''^3x'''''-45 x'''^3x^{13}x''^2x''''+$
$36 x^{11}x''^5x'^2x'''''+ 72 x^{10}x''^5x'^3x'''', r^2a'+ 2 arr', x^2a^2-$
$x^2x''^2- x''^2y^2, r^2- x^2- y^2\}.$

Example 5.2: [11]. We take the following d-polynomial set$\mathbb{T}$ consisting of three d-polynomials T_1, T_2 and T_3.

$$\mathbb{T} = \{x'- x^2-xy, y'+ yx+ y^2, x'^2 + y'^2 + z'^2-1\},$$

with$x \prec y \prec z$.

Compute a d-char set of the set

$$\mathbb{T} = \{T_1, T_2, T_3\},$$

by the algorithm NewdCharSet, one gets following d-polynomial set as the output:

$\{x'- x^2-xy, y'+ yx+ y^2,-1 + z'^2 + yx^2 + 2yxy^2 +$
$y^4 + x^4 + 2x^2xy + xy^2\}.$

Example 5.3: [11]. Take five d-polynomials (the pendulum in Cartesian coordinates) and form a set $\mathbb{R}$ consisting of five

d-polynomials:$\mathbb{R} = [R_1, \ldots, R_5]$

$R_1 = mq'+ \lambda y+ g$, $R_2 = y'- q$, $R_3 = mp'+ \lambda x$,$R_4 = x'- p$, $R_5 = x^2 + y^2-1$,

with $\lambda \prec x \prec y \prec p \prec q$..

Let us compute a d-char set of d-polynomial set

$\mathbb{R} = [R_1, R_2, R_3, R_4, R_5]$,

by the algorithm NewdCharSet, we get the following d-charset in the output:

$\{x^6\lambda^5g^{10}-5x^6\lambda^7g^8 + 8x^6\lambda^9g^6-3x^4\lambda^5g^{10} + 15x^4\lambda^7g^8-24x^4\lambda^9g^6-5x^6\lambda^{11}g^4 + 15x^4\lambda^{11}g^4 + 3x^2\lambda^5g^{10}-15x^2\lambda^7g^8+24x^2\lambda^9g^6-15x^2\lambda^{11}g^4+\lambda^{13}g^2x^6+3\ \lambda^{13}g^2x^2-3\ \lambda^{13}g^2x^4+5\ g^8\lambda^7-g^{10}\lambda^5-8\ g^6\lambda^9+5\ g^4\lambda^{11}-\lambda^{13}g^2, mp'+\lambda x, mq'+\lambda y+g, x^2+y^2-1, -x'+p, yq+xx'\}$.

Example 5.4: [11]. Take a d-polynomial set $\mathbb{S}$ consisting ofseven d-polynomials $S_1, S_2, S_3, S_4, S_5, S_6$and S_7 defined asfollows:

$$\mathbb{S} = \{x'^1-p^1x_1^2-p_2x_1x_2-u,\ x_2'-p_3x_1^2+p_4,\ y-x_1,\ p_1', \ldots, p_4'\}$$

with $u \prec y \prec p_4 \prec p_3 \prec p_2 \prec p_1 \prec x_2 \prec x_1$.
If one computes a d-char set of the set

$$\mathbb{S} = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\},$$

by NewdCharSet algorithm, then one gets the following d-charset:

$$\{p_1', p_2', p_3', p_4', -y+x^1, -x_2''-p_4+y^2p_3, u+y\,p_2x^2+y^2p_1\}.$$

Example 5.5: [19]. If a particle moves in a plane under a central force which is proportional to the radius drawn from the particle to the force center. We compute its d-char set.
Let the coordinates of the particle be $(x(t), y(t))$ where $t$istime. We assume that the force center is the origin point $(0,0)$.Let $a$be the magnitude of the acceleration of the particle and$r$be the length of the radius vector drawn from the particle tothe force center. The problem can be represented as following equations (see details [19])

$H_1 = r^2 - x^2 - y^2 = 0,$
$H_2 = a^2 - x''^2 - y''^2 = 0,$
$H_3 = x''y - y''x = 0$, the force is toward the origin,
$H_4 = ld(a, r) = a'r - r'a$, a is proportional to r.
Take a d-polynomial set

$$\mathbb{H} = \{H_1, H_2, H_3, H_4\}$$

with ordering $x \prec y \prec r \prec a$. Compute a d-char set of thed-polynomial set $\mathbb{H}$ by the NewdCharSet algorithm. We getthe following d-char set:
$\{x''(-2\ x^2x''\ x'\ y^2 + x'''\ x^5 + 2x'''\ x^3y^2 - x^4x''\ x' + xx'''\ y^4 - x''\ y^4x')x^2x''2 - x''^2y^2, r^2 - x^2 - y^2\}$

The Table 1 provides information about the test examples.
We have the following information from Table 1;
nops: the number terms in each d-polynomial which appears in d-char set of each example.
time: the time in seconds for computing d-char set for each example.
For example, the entry 64, 3, 4, 3 in the second column and first row represent the obtained d-char set consists of aset of four d-polynomials having 64, 3, 4, 3 number of terms respectively. Similarly the entry 12.808 s in first row and third column represents that the d-char set of the d-polynomial set$\mathbb{F}$ is computed in 0.132s by the algorithm NewdCharSet.

Table I
Nops And Timing In The Outputs Of Newdcharset

No	Nops	Time
1	64, 3, 4, 3	12.808s
2	3, 3, 8	0.016s
3	20, 2, 3, 3, 2, 2	0.140s
4	1, 1, 1, 1, 2, 2, 3	0.47s
5	6, 3, 3	0.047s

The Table 2 collects information about **degree tuple.** The degree tuple of each d-polynomial obtained in the output of d-char set computed by NewdCharSet. Let us explain the meanings of the entries in table 2, for example the entry in the second row and the second column containing three tuples implies that the output of NewdCharSet for example 2 consists of three d-polynomials, say F_1, F_2, F_3,the degrees of F_1 in x, y, z ($x \prec y \prec z$) are 3, 3, 8 respectively.

Table Ii
Deg Tuple Of D-Polynomial Set In The Output Of Newdcharset

No	Degree Tuple
1	[13, 6, 0, 0], [2, 2, 2, 0], [2, 2, 0, 1], [2, 2, 0, 2]
2	[2,0,0], [0,2,0], [4,4,0]
3	[13,6,0,0,0], [1,1,0,0,0], [1,0,1,0,0], [0,2,2,0,0], [0,0,0,1,0], [0,1,1,0,1]
4	[0,0,0,0,0,0,0], [0,0,0,0,0,0,0], [0,0,0,0,0,0,0], [0,0,0,0,0,0,0], [0,1,0,0,0,1,0], [0,2,0,0,1,1,0], [1,2,0,0,1,1,0]
5	[5, 4, 0, 0],[2, 2, 0, 2],[2, 2, 2, 0]

6. REMARKS AND FUTURE WORK

A necessary condition for C to be a d-char set of a d-polynomial set P is that all d-polynomials in P have d-pseudo-remainders 0 w.r.t C. This condition is weakened now by replacing the d-polynomial P with an arbitrary d-polynomial set that generates the same ideal as the d-polynomial P, which leads to generalize the d-char sets for the ordinary d-polynomial sets. In this way the expression swell of immediate d-polynomials in the process of variable elimination by means of d-pseudo-division can be avoided and we get simpler
outputs for large problems.
In present paper the concept of generalized d-char sets has been presented for ordinary d-polynomial sets. Comparison with the existing schemes for computing d-char sets for ordinary d-polynomial sets and their physical applications will be in our consideration further. This concept will be extended for d-polynomial systems in forthcoming papers. It will be interesting to extend this idea to the partial differential case as well.
Admissible d-reductions which are used in generalized d-char sets may also be applied to other algorithms for triangular decomposition. Moreover, the condition of admissible d-reduction is quite flexible, other kinds of d-reductions can also be imported, for example modular relation between d-polynomials and techniques from linear algebra.

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