A NOTE ON THE RGULARITY OF A CLASS OF STABLE IDEALS

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ABSTRACT. In this paper, we give an upper bound of the Castelnuovo-Mumford regularity of a class of stable ideals. We discuss those stable ideals, whose associated prime ideals are totally ordered under inclusion and whose irreducible primary decomposition consists the ideals of the type $(x_1^{\alpha_1},\dots,x_r^{\alpha_r})$ *with* $r\geq 3$ *and* $r-1\geq\alpha_1\geq\cdots\geq\alpha_r=1$. *We name these ideals as* α*-ideals.*

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1. INTRODUCTION

Let $S=K[x_1,\dots,x_n],n\geq 2$ be the polynomial ring over the infinite field K and $I\subset S$ a monomial ideal. Let $G(I)$ be the minimal set of monomial generators of I and $\deg(I)$ the highest degree of a monomial of $G(I)$. Given a monomial $u\in S$ set $m(u)=\max\{i : x_i|u\}$ and $m(I)=\max_{u\in G(I)} m(u)$. If $\beta_{ij}(I)$ are the graded Betti numbers of I then the regularity of I is given by $\text{reg}(I)=\max\{j-i|\beta_{ij}(I)\neq 0\}$.

Set $q(I)=m(I)(\deg(I)-1)+1$. In [4, Remark 2.5] authors sowed that $\text{reg}(I)\leq q(I)$ for a homogeneous ideal I of height n (clearly, $m(I)=n$ in this case). Bayer and Mumford [2], Caviglia and Sbarra [4] and Mayr and Meyer [8] showed that the regularity of a homogeneous ideal could grow exponentially with respect to its degree. A monomial ideal I is stable if for each monomial $u\in I$ and $1\leq j<m(u)$ it follows $x_ju/x_{m(u)}\in I$. If I is so called p-Borel ideal then $\text{reg}(I)\leq n\deg(I)$ as Popescus proved in [9]. Ahmad, Anwar [1] and Cimpoeas [5] showed that for a monomial ideal I whose associated prime ideals are totally ordered under inclusion, $\text{reg}(I)\leq q(I)$.

Let $I\subset S=k[x_1,x_2,\dots,x_n]$ be a monomial ideal with irreducible primary decomposition $I=\cap_{i=1}^{s}Q_i$ where each Q_i is of the form; $Q_i=(x_1^{\alpha_1},x_2^{\alpha_2},\dots,x_{r_i}^{\alpha_{r_i}})$ with $r_i\geq 3$ and $r_i-1\geq\alpha_1\geq\cdots\geq\alpha_{r_i}=1$. We call such ideals as α*-Ideals*.

For more detail about irreducible ideals and irredundant irreducible primary decomposition of a monomial ideal see ([10, Theorem 5.1.17]). In this paper we give an upper bound on the regularity of α-ideals.

We should mention that our presentation was improved by the kind suggestions of any anonymous Referee.

2. Stability of the α-Ideals

Let k be an infinite field. Let $S=k[x_1,\dots,x_n],n\geq 2$ be the polynomial ring over k and $I\subset S$ a monomial ideal. Let $I_{\geq\tilde{q}(I)}$ be the ideal generated b the monomials of I of degree $\geq\tilde{q}(I)$.

Definition 2.1 A monomial ideal $I\subset S$ with

$$I=\bigcap_{i=1}^{s}(x_1^{\alpha_1},\dots,x_{r_i}^{\alpha_{r_i}})$$

is said to be an α-ideal if

(i) $3\leq r_1<r_2<\cdots<r_s=n$ and

(ii) $r_i-1\geq\alpha_1\geq\cdots\geq\alpha_{r_i}=1$ for all $1\leq i\leq s$.

Example 2.2 Let $S=k[x_1,x_2,x_3,x_4,x_5,x_6]$, then the monomial ideal

$I=(x_1,x_2,x_3)\cap(x_1^2,x_2^2,x_3,x_4)\cap(x_1^3,x_2^3,x_3^3,x_4^2,x_5,x_6)$ is an α-ideal.

Definition2.3 Let $I=\cap_{i=1}^{s}Q_i\subset S$ be an α-ideal with $Q_i=\left(x_1^{\alpha_1},x_2^{\alpha_2},\dots,x_{r_i}^{\alpha_{r_i}}\right)$, then we define $\tilde{q}(I)$ as follows:

$\tilde{q}(I)=\max\{\tilde{q}(Q_i)|1\leq i\leq s\}$.

Where, we set $\tilde{q}(Q_i)=\sum_{i=2}^{r}\alpha_i$.

Proposition 2.4 The monomial ideal $I=\left(x_i^{\alpha_1},\dots,x_r^{\alpha_r}\right)\subset S$ has $I_{\tilde{q}(I)}$ stable for $r\in\{3,\dots,n\}$ and $r-1\geq\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_r=1$, where $\tilde{q}(I)=\sum_{i=2}^{r}\alpha_i$.

Proof Let $u\in I_{\geq\tilde{q}(I)}$ be a monomial. From above we get $u=v.x_j^{\alpha_j}$ for some $1\leq j\leq r$ and $v\in(x_1,\dots,x_n)^{\tilde{q}-\alpha_j}$. If $m(u)>j$, then $\frac{x_k}{x_{m(u)}}.u=\left(\frac{x_k.v}{x_{m(u)}}\right).x_j^{\alpha_j}\in I_{\geq\tilde{q}(I)}$ for all $k<m(u)$.

If $m(u)=j$ then u belongs to the stable ideal $(x_1,\dots,x_n)^{\tilde{q}(I)}$ and it is enough to show that $(x_1,\dots,x_n)^{\tilde{q}(I)}\subset I_{\geq\tilde{q}(I)}$. Let $w\in(x_1,\dots,x_n)^{\tilde{q}(I)}$ then $w=x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ with all $a_i\geq 0$ and $\sum_{i=1}^{n}a_i\geq\tilde{q}(I)=\sum_{i=2}^{r}\alpha_i$. Now we will prove that there exist some k such that $1\leq k\leq n$ with $a_k\geq\alpha_k$. Suppose that there does not exist such k, that is $a_i<\alpha_i$ for all $i\in\{1,\dots,n\}$. Therefore from the above summations

$$a_i\geq(\alpha_2-a_2)+(\alpha_3-a_3)+\cdots+(\alpha_n-a_n)$$

As $a_i>\alpha_i$, so we have $\alpha_1\geq n-1$, it implies $\alpha_1>n-1$, which is a contradiction.

So we can take above $v=x_1^{a_1}\cdots x_k^{a_k-\alpha_k}\cdots x_n^{a_n}$.

Remarks 2.5 In general one cannot get $I_{\tilde{q}(I)-1}$ stable, when $I=(x_1^{\alpha_1},x_2^{\alpha_2},\dots,x_r^{\alpha_r})$ with $n-1\geq\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_r=1$. For example, for $r=3$ and $I=(x_1^2,x_2^2,x_3)$ then $\tilde{q}(I)=3$ and clearly $I_{\geq 2}$ is not stable.

Now we recall the following results from [1].

Proposition 2.6 [1] If I,J are monomial ideals such that $I_{q(I)}$ and $J_{\geq q(J)}$ are stable ideals, then $(I\cap J)_{\geq\max(q(I),q(J))}$ is stable.

Lemma2.7 Let I is an α-ideal in the polynomial ring $S = K[x_1, x_2, \dots, x_n]$ with $n \geq 2$. Then $I_{\geq \tilde{q}(I)}$ is stable.

Proof As I is an α-ideal in S. So, the irreducible irredundant primary decomposition of I is;

$$I = \bigcap_{i=1}^{s} Q_i,$$

where each $Q_i = (x_i^{\alpha_1}, \dots, x_i^{\alpha_{r_i}})$ with $3 \leq r_i \leq n$ for all $i \in \{1, 2, \dots, s\}$. Hence from 2.6, it immediately follows that $I_{\geq \tilde{q}(I)}$ is stable.

Next we recall a proposition from [6].

Proposition 2.8 Let I be a monomial ideal and $e \geq deg(I)$ an integer such that $I_{\geq e}$ is stable. Then $reg(I) \leq e$.

Theorem 2.9 Let $I \in S$ be an α-ideal. Then $reg(I) \leq \tilde{q}(I)$.

Proof By the previous Lemma 2.7, we have $I_{\geq \tilde{q}(I)}$ stable, as $\tilde{c}(I) \geq deg(I)$. Hence we get $reg(I) \leq \tilde{q}(I)$ by proposition 2.8.

Remarks2.10 It should be noted that α-ideal is a monomial ideal whose associated prime ideals are totally ordered under inclusion. In [1] and [6], the authors have given the bound for the regularity of such ideals that is $reg(I) < g(I) = m(I)(deg(I) - 1)$+1. Therefore, we have $reg(I) \leq q(I)$ for α-ideals I. But it is worth noting that our bound $\tilde{q}(I) \leq q(I)$, for instance the α-ideals $I = (x_1, x_2) \cap (x_1^2, x_2^2, x_3)$, $\tilde{q}(I) = 3 < q(I) = 4$.

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