ON HEMICONTRACTIONS IN HILBERT SPACES

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Abstract In this note, we establish the strong convergence for the Ishikawa iterative scheme with errors associated with Lipschitzian pseudocontractive mappings in Hilbert spaces.

Key Words Ishikawa iterative scheme with errors, Lipschitzian mappings, Pseudocontractive mappings, Hilbert spaces

INTRODUCTION

Let H be a Hilbert space. A mapping $T: H \to H$ is said to be *pseudocontractive* (see for example, [1, 2]) if

$$\| Tx - Ty \|^2 \leq \| x - y \|^2 + \| (I-T)x - (I-T)y \|^2, \quad \forall x, y \in H \tag{1.1}$$

and is said to be *strongly pseudocontractive* if there exists $k \in (0,1)$ such that

$$\| Tx - Ty \|^2 \leq \| x - y \|^2 + k \| (I-T)x - (I-T)y \|^2, \forall x, y \in H. \tag{1.2}$$

Let $F(T) := \{x \in H : Tx = x\}$ and let K be a nonempty subset of H. A mapping $T: K \to K$ is called *hemicontractive* if $F(T) \neq \varnothing$ and

$$\| Tx - x^* \|^2 \leq \| x - x^* \|^2 + \| x - Tx \|^2$$

$$\forall\, x \in H, x^* \in F(T).$$

It is easy to see that the class of pseudocontractive mappings with fixed points is a subclass of the class of hemicontractions. The following example, due to Rhoades [6], shows that the inclusion is proper. For $x \in 0,1]$, define $T:[0,1] \to 0,1]$ by $Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$. It is shown in [6] that T is not Lipschitz and so cannot be nonexpansive. A straightforward computation (see for example, [7]) shows that T is pseudocontractive. For the importance of fixed points of pseudocontractions the reader may consult [1].

In 1974, Ishikawa [4] proved the following result:

Theorem 1 *If K is a compact convex subset of a Hilbert space H, $T: K \mapsto K$ is a Lipschitzian pseudocontractive map and x_0 is any point in K, then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer $n \geq 1$ by*

$$x_{n+1} = (1 - \sim_n) x_n + \sim_n T y_n,$$

$$y_n = (1 - \ _n) x_n + \ _n T x_n,$$

where $\{\sim_n\}, \{\ _n\}$ are sequences of positive numbers satisfying the conditions

$$(i) 0 \leq \sim_n \leq \ _n < 1; (ii) \lim_{n \to \infty} \ _n = 0; (iii) \sum_{n \geq 1} \sim_n \ _n = \infty. \tag{1.4}$$

Another iteration scheme which has been studied extensively in connection with fixed points of pseudocontractive mappings is the following: For K a convex subset of a Banach space E, and $T: K \to K$, the sequence $\{x_n\}$ is defined iteratively by $x_1 \in K$,

$$x_{n+1} = (1 - c_n) x_n + c_n T x_n, n \geq 1, \tag{1.5}$$

where $\{c_n\}$ is a real sequence satisfying the following conditions:

$$(iv)\, 0 \leq c_n < 1; (v) \lim_{n \to \infty} c_n = 0; (vi) \sum_{n=1}^{\infty} c_n = \infty. \tag{1.6}$$

The iteration scheme (1.5) is generally referred to as the *Mann iteration process* in light of [5].

In 1997, Xu [8] introduced the following iteration scheme: Let K be a nonempty convex subset of a Banch space E and $T: K \to K$ a mapping. For any given $x_1 \in K$, the sequence $\{x_n\}$ defined iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n,$$

$$y_n = a_n' x_n + b_n' T x_n + c_n' v_n, n \geq 1,$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in K and $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\}$ and $\{c_n'\}$ are sequences in [0,1] such that $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1$ for all $n \geq 1$ is called the *Ishikawa iteration sequence with errors in the sense of Xu*.

If, with the same notations and definitions as in (1.7), $b_n' = c_n' = 0$, for all integers $n \geq 1$, then the sequence $\{x_n\}$ now defined by

$$x_1 \in K, \tag{1.8}$$

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, n \geq 1,$$

is called the *Mann iteration scheme with errors in the sense of Xu*.

We remark that if K is bounded (as is generally the case), (1.3) the error terms u_n, v_n are *arbitrary* in K.

In [3], Chidume and Chika Moore generalized the results of Ishikawa for continuous pseudocontractions and proved the following results.

Theorem 2 *3] Let K be a compact convex subset of a real Hilbert space H; $T: K \to K$ a continuous hemicontractive mapping. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\}$*

and $\{c_n'\}$ *be real sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $a_n + b_n + c_n = 1 = a_n' + b_n' + c_n'$ for all $n \ge 1$;

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = 0$;

(iii) $\sum c_n < \infty; \sum c_n' < \infty$;

(iv) $\sum \sim_n \ _n = \infty; \sum \sim_n \ _n ¢_n < \infty, where$ $¢_n := \| Tx_n - Ty_n \|^2$;

(v) $0 \le \sim_n \le \ _n < 1 \ \forall\, n \ge 1, where$ $\sim_n := b_n + c_n; \ \ _n := b_n' + c_n'.$

For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n,$$
$$y_n = a_n' x_n + b_n' T x_n + c_n' v_n, n \ge 1,$$

where $\{u_n\},\{v_n\}$ are arbitrary sequences in K. Then, $\{x_n\}$ converges strongly to a fixed point of T.

Remark 1 *For the proof of Theorem 2, Chidume and Moore used the condition* $\sim_n \ _n \|Tx_n - Ty_n\|^2 < \infty$. *Since* K *is compact, so for some constant* $M' \ge 0$, *we obtain* $\sim_n \ _n \|Tx_n - Ty_n\|^2 = \infty$. *Hence the problem is still open.*

In this paper, we establish the strong convergence for the Ishikawa iterative scheme with errors associated with Lipschitzian pseudocontractive mappings in Hilbert spaces.

Preliminaries

We shall make use of the following well known results.

Lemma 1 *8] Suppose that* $\{°_n\},\{±_n\}$ *are two sequences of nonnegative numbers such that for some real number* $N_0 \ge 1$,

$$°_{n+1} \le °_n + ±_n \ \forall n \ge N_0.$$

(a) If $\sum ±_n < \infty$, then, $\lim °_n$ exists.

(b) If $\sum ±_n < \infty$ and $\{°_n\}$ has a subsequence converging to zero, then $\lim °_n = 0$.

Lemma 2 *10] For all* $x, \ y \in H$ *and* $ª \in 0,1]$, *the following well-known identity [17] holds:*

$$\|(1-ª)x + ªy\|^2 = (1-ª)\|x\|^2 + ª\|y\|^2 - ª(1-ª)\|x-y\|^2$$

Lemma 3 *Let* H *be a Hilbert space, then for all* $x, y, z \in H$

$$\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2$$
$$- bc\|y-z\|^2 - ca\|z-x\|^2,$$

where $a,b,c \in 0,1]$ and $a+b+c=1.$

Main Results

Now we prove our main results.

Theorem 3 *Let* K *be a compact convex subset of a real Hilbert space* H *;* $T : K \to K$ *a Lipschitzian hemicontractive mapping satisfying*

$$\|x - Ty\| \le \|Tx - Ty\| \text{ for all } x, y \in K. \qquad \text{(C)}$$

Let $\{a_n\},\{b_n\},\{c_n\},\{a_n'\},\{b_n'\}$ and $\{c_n'\}$ be real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = 1 = a_n' + b_n' + c_n'$ for all $n \ge 1$;

(ii) $\sum c_n < \infty; \sum c_n' < \infty$;

(iii) $\sum b_n b_n' = \infty$;

(iv) $\lim_{n\to\infty} b_n' = 0 \ \forall\, n \ge 1.$

For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n,$$
$$y_n = a_n' x_n + b_n' T x_n + c_n' v_n, n \ge 1,$$

where $\{u_n\},\{v_n\}$ are arbitrary sequences in K. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. From Schauder's fixed point theorem, $F(T)$ is nonempty (where $F(T)$ denotes the set of fixed points of T) since K is a convex compact set and T is continuous, let $x^* \in F(T)$. Set $M = 1 + diam(K)$. Using the fact that T is hemicontractive we obtain

$$\|Tx_n - x^*\|^2 \le \|x_n - x^*\|^2 + \|x_n - Tx_n\|^2, \qquad (3.1)$$

and

$$\|Ty_n - x^*\|^2 \le \|y_n - x^*\|^2 + \|y_n - Ty_n\|^2. \qquad (3.2)$$

With the help of (1.7), (3.1), (3.2) and Lemma 3, we obtain the following estimates:

$$\|y_n - x^*\|^2 = \|a_n' x_n + b_n' Tx_n + c_n' v_n - x^*\|^2$$
$$= \|a_n'(x_n - x^*) + b_n'(Tx_n - x^*) + c_n'(v_n - x^*)\|^2$$
$$= a_n'\|x_n - x^*\|^2 + b_n'\|Tx_n - x^*\|^2 + c_n'\|v_n - x^*\|^2$$
$$- a_n' b_n'\|x_n - Tx_n\|^2 - b_n' c_n'\|Tx_n - v_n\|^2$$
$$- a_n' c_n'\|x_n - v_n\|^2$$
$$\le (1 - b_n')\|x_n - x^*\|^2 + b_n'\left(\|x_n - x^*\|^2 + \|x_n - Tx_n\|^2\right) + M^2 c_n'$$
$$- a_n' b_n'\|x_n - Tx_n\|^2$$
$$= \|x_n - x^*\|^2 + b_n'(1 - a_n')\|x_n - Tx_n\|^2 + M^2 c_n',$$
$$\|y_n - Ty_n\|^2 = \|a_n' x_n + b_n' Tx_n + c_n' v_n - Ty_n\|^2$$
$$= \|a_n'(x_n - Ty_n) + b_n'(Tx_n - Ty_n) + c_n'(v_n - Ty_n)\|^2$$

$$= a_n'\|x_n - Ty_n\|^2 + b_n'\|Tx_n - Ty_n\|^2 + c_n'\|v_n - Ty_n\|^2$$
$$- a_n'b_n'\|x_n - Tx_n\|^2 - b_n'c_n'\|Tx_n - v_n\|^2$$
$$- a_n'c_n'\|x_n - v_n\|^2$$
$$\le a_n'\|x_n - Ty_n\|^2 + b_n'\|Tx_n - Ty_n\|^2 + M^2 c_n'$$
$$- a_n'b_n'\|x_n - Tx_n\|^2.$$

Substituting (3.3) and (3.4) in (3.2) we obtain

$$\|Ty_n - x^*\|^2 \le \|x_n - x^*\|^2 + a_n'\|x_n - Ty_n\|^2 + b_n'\|Tx_n - Ty_n\|^2$$

(3.5)

$$- b_n'(2a_n' - 1)\|x_n - Tx_n\|^2 + 2M^2 c_n'.$$

Also with the help of (3.5), we have

$$\|x_{n+1} - x^*\|^2 = \|a_n x_n + b_n Ty_n + c_n u_n - x^*\|^2$$
$$= \|a_n(x_n - x^*) + b_n(Ty_n - x^*) + c_n(u_n - x^*)\|^2$$
$$= a_n\|x_n - x^*\|^2 + b_n\|Ty_n - x^*\|^2 + c_n\|u_n - x^*\|^2$$
$$- a_n b_n\|x_n - Ty_n\|^2 - b_n c_n\|Ty_n - u_n\|^2 - a_n c_n\|x_n - u_n\|^2$$
$$\le (1 - b_n)\|x_n - x^*\|^2$$
$$+ b_n\Big(\|x_n - x^*\|^2 + a_n'\|x_n - Ty_n\|^2 + b_n'\|Tx_n - Ty_n\|^2$$
$$- b_n'(2a_n' - 1)\|x_n - Tx_n\|^2 + 2M^2 c_n'\Big)$$
$$+ M^2 c_n$$
$$= \|x_n - x^*\|^2 + b_n b_n'\|Tx_n - Ty_n\|^2$$
$$- b_n b_n'(2a_n' - 1)\|x_n - Tx_n\|^2 - b_n(a_n - a_n')\|x_n - Ty_n\|^2$$
$$+ M^2 c_n + 2M^2 b_n c_n'$$
$$\le \|x_n - x^*\|^2 + b_n b_n'\|Tx_n - Ty_n\|^2$$
$$- b_n b_n'(1 - 2(b_n' + c_n'))\|x_n - Tx_n\|^2 - b_n(b_n' - b_n)\|x_n - Ty_n\|^2$$
$$+ 2M^2(c_n + c_n')$$
$$\le \|x_n - x^*\|^2 + b_n b_n'\|Tx_n - Ty_n\|^2 - b_n b_n'(1 - 2(b_n' + c_n'))\|x_n - Tx_n\|^2$$
$$+ b_n^2\|x_n - Ty_n\|^2$$
$$\le \|x_n - x^*\|^2 + b_n(b_n' + b_n)\|Tx_n - Ty_n\|^2 - b_n b_n'(1 - 2(b_n' + c_n'))\|x_n - Tx_n\|^2$$
$$\le \|x_n - x^*\|^2 + 2b_n\|Tx_n - Ty_n\|^2 - b_n b_n'(1 - 2(b_n' + c_n'))\|x_n - Tx_n\|^2.$$

Observe that

$$\|x_n - y_n\| = \|x_n - a_n'x_n - b_n'Tx_n - c_n'v_n\|$$
$$= \|b_n'(x_n - Tx_n) + c_n'(x_n - v_n)\|$$
$$\le b_n'\|x_n - Tx_n\| + c_n'\|x_n - v_n\|,$$

and since T is Lipschitzian,

$$\|Tx_n - Ty_n\|^2 \le L^2\|x_n - y_n\|^2$$
$$\le L^2\left(b_n'\|x_n - Tx_n\| + Mc_n'\right)^2$$
$$\le L^2 b_n'^2\|x_n - Tx_n\|^2 + 3L^2 M^2 c_n',$$

and consequently from (3.6) we obtain

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2$$
$$- b_n b_n'(1 - 2(1 + L^2)b_n' - 2c_n')\|x_n - Tx_n\|^2 + 2M^2 c_n + 2M^2(1 + 3L^2)c$$

(3.6)

Now by $\lim_{n\to\infty} b_n' = 0 = \lim_{n\to\infty} c_n'$, imply that there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$b_n' \le \frac{1}{6(1 + L^2)} \text{ and } c_n' \le \frac{1}{6}, \quad (3.8)$$

and from (3.7) we get

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - \frac{2}{3} b_n b_n'\|x_n - Tx_n\|^2$$
$$+ 2M^2 c_n + 2M^2(1 + 3L^2)c_n',$$

implies

$$\frac{2}{3} b_n b_n'\|x_n - Tx_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2M^2 c_n + 2M^2(1 + 3$$

Thus

$$\frac{2}{3}\sum_{j=0}^{\infty} b_j b_j'\|x_j - Tx_j\|^2 \le \sum_{j=0}^{\infty} \text{¢}_j +$$
$$\sum_{j=0}^{\infty}\left(\|x_j - x^*\|^2 - \|x_{j+1} - x^*\|^2\right)$$
$$= \sum_{j=0}^{\infty} \text{¢}_j + \|x_0 - x^*\|^2,$$

where $\text{¢}_j = 2M^2 c_j + 2M^2(1 + 3L^2)c_j'$.

Hence by conditions (ii) and (iii), we have

$$\sum_{j=0}\|x_{j-1} - Tv_j\|^2 < +\infty. \quad (3.10)$$

It implies that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

The rest of the argument follows exactly as in the proof of Theorem 1 of [3] and the proof is complete.

Theorem 4 *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive mapping satisfying*

$$\|x - Ty\| \le \|Tx - Ty\| \text{ for all } x, y \in K.$$

Let $\{a_n\},\{b_n\},\{c_n\},\{a_n'\},\{b_n'\}$ and $\{c_n'\}$ be real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = 1 = a_n' + b_n' + c_n'$ for all $n \geq 1$;

(ii) $\sum c_n < \infty; \sum c_n' < \infty$;

(iii) $\sum b_n b_n' = \infty$;

(iv) $\lim_{n\to\infty} b_n' = 0 \; \forall \; n \geq 1$.

Let $P_K : H \to K$ be the projection operator of H onto K. Then the sequence $\{x_n\}$ defined iteratively by

$$x_{n+1} = P_K\left(a_n x_n + b_n T y_n + c_n u_n\right),$$

$$y_n = P_K(a_n' x_n + b_n' T x_n + c_n' v_n), n \geq 1,$$

where $\{u_n\}$ and $\{u_n\}$ are arbitrary sequences in K, converges strongly to a fixed point of T.

Proof. The operator P_K is nonexpansive (see e.g., [2]). K is a Chebyshev subset of H so that, P_K is a single-valued mapping. Hence, we have the following estimate:

$$\|x_{n+1} - x^*\|^2 = \|P_K(a_n x_n + b_n T y_n + c_n u_n) - P_K x^*\|^2$$
$$\leq \|a_n x_n + b_n T y_n + c_n u_n - x^*\|^2$$
$$= \|a_n (x_n - x^*) + b_n (T y_n - x^*) + c_n (u_n - x^*)\|^2$$
$$\leq \left\|x_n - x^*\right\|^2 - b_n b_n'\left(1 - 2(1+L^2) b_n' - 2c_n'\right)\left\|x_n - T x_n\right\|^2$$
$$+ 2M^2 c_n + 2M^2\left(1 + 3L^2\right)c_n'.$$

The set $K = K \cup T(K)$ is compact and so the sequence $\{\|x_n - T x_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 3 and the proof is complete.

Remark 2 *This kind of reconstruction for Lipschitz hemicontractive mappings is new under the setting of Hilbert spaces.*

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