EVALUATION OF OVERCOMPLETE DICTIONARIES FOR IMAGE INPAINTING

Abdul Qayyum[#1], Aamir Saeed Malik[#2], Mohamad Naufal Mohamad Saad[#3], Mohamad Faris Abdullah[#4], Mahboob Iqbal[#5]

#Centre for Intelligent Signal and Imaging Research (CISIR), Electrical & Electronic Engineering Department
Universiti Teknologi PETRONAS, Bandar Seri Iskandar, 31750 Tronoh, Perak, MALAYSIA
Email: engr.qayyum@gmail.com

ABSTRACT*—In this paper, we present the design of over-complete dictionaries based on the optimization algorithm using sparse representation. In addition, optimization of the proposed dictionaries using KSVD algorithm is performed. We proposed three dictionaries based on optimization greedy algorithms like basis pursuit (BP) and orthogonal matching pursuit (OMP). Furthermore, the performance of dictionaries is computed involving OMP and BP. The result shows that the basis pursuit greedy algorithm performs better and also the dictionary based on KSVD algorithm performs well as compared to the fixed dictionaries like discrete wavelet transform (DWT), Discrete cosine transform (DCT) as well as discrete Tchebichef transform (DTT). The DDT is used as a basis for the first time to store the image patch as a dictionary. The comparison was made between dictionaries needs to apply the sparse representation for many applications. After applying the greedy algorithms, the result shows that the BP produced less error as compared to OMP based on our proposed dictionaries with less increase in the computational complexity. Results showed that the KSVD based dictionary performs well for image in-painting application.*

Keywords—Over-complete dictionaries, DCT, DWT, DTT, KSVD, SpareRepresentation.

1. INTRODUCTION

The sparse representation has a wide range of applications in image processing and computer vision such as image separation using sparse representation, Image in-painting, signal source separation, image classification, de-noising, signal blind source separation. In recent years sparse representations have much attention due to their wide variety of applications used in image processing. In sparse representation the problem can be solved for the most compressed representation of a signal in terms of a linear combination of atoms in an over-complete dictionary. The process of obtaining a sparse representation of a signal or image requires explicit knowledge of the synthesis dictionary. One crucial problem in a sparse representation based application is how to choose the dictionary [1]. Now days the representation of signals in multi-orientation and multiscale there are many transforms used such as ridgelet, curvelet, counterlet, shearlet, bandlet. These are the motivation for the research on the sparse representation. Sparse representation produces better performance as compared to other methods based on time domain processing and orthonormal basis transforms. Particularity, the focus of research based on sparse representation has three parts. First is the dictionary design and trained dictionary using well known KSVD algorithm, the second and most important is optimization algorithms based on pursuit methods like matching pursuits, orthogonal matching pursuit and basis pursuits and third area is the LARS/homotopy methods [4]. We are implementing the fixed dictionaries to measure the sparse representation and applied for the most important image application is in-painting the image using sparse representation based on different fixed dictionaries as well as an adaptive KSVD dictionary.

There are two types of dictionaries used in sparse representation that are predetermined and adaptive dictionaries. The predetermined dictionaries are curvelet, discrete cosine, wavelet, ridgelet and bandlet, and adaptive dictionaries are trained KSVD and Method of directions (MOD) algorithms. The over-complete dictionary is generated by combining multiple standard transforms, including curvelet transform, ridgelet transform and discrete cosine transform. The over-complete predetermined dictionary such as wavelet based dictionary, a shape based and region based dictionary are used to represent the small image patch sparsely [6]. Currently the most mostly used transforms for execution that task are the Discrete Cosine Transform (DCT) and Discrete Wavelet Transforms (DWT). An important reason for the attractiveness of both these transforms is the feasibility of their fast implementation. Wavelet and DCT transforms are broadly used in image processing applications. Many image processing tasks take advantage of sparse representations of image data where most information is packed into a small number of samples. Typically, these representations are achieved via invertible and non-redundant transforms. Despite the fact that wavelets have had a wide impact in image processing, they fail to efficiently represent objects with highly anisotropic elements such as lines or curvilinear structures (e.g. edges). The reason is that wavelets are non-geometrical and do not exploit the regularity of the edge curve. The success of wavelets is mainly due to the good performance of piecewise smooth functions in one dimension. Unfortunately, such is not the case in two dimensions. In essence, wavelets are good at catching zero-dimensional or point singularities, but two-dimensional piecewise smooth signals resembling images have one-dimensional singularities. That is, smooth regions are separated by edges, and while edges are discontinuous across, they are typically smooth curves. Intuitively, wavelets in two dimensions are obtained by a tensor-product of one dimensional wavelets and they are thus good at isolating the discontinuity across an edge, but will not see the smoothness along the edge.

In order to use over-complete and sparse representations in applications, one need to fix a dictionary D, and then find efficient ways to solve (1). Exact determination of sparsest representations proves to be an NP-hard problem [1]. Hence, approximate solutions are considered instead. In the last decade or so, several efficient pursuit algorithms have been proposed. The simplest ones are the Matching Pursuit (MP)

[2] or the Orthogonal Matching Pursuit (OMP) algorithms [3]. These are greedy algorithms that select the dictionary atoms sequentially. A second well known pursuit approach is the Basis Pursuit (BP) [4]. The algorithms range widely in empirical effectiveness, computational cost, and implementation complexity. Unfortunately, there is little guidance available on choosing a good technique for a given parameter regime [5]. Extensive study of these algorithms in recent years has established for the solution is sparse enough [6].

Two types of dictionaries are used in sparse representation: predetermined and adaptive. The predetermined dictionaries are curvelet, wavelet ridgelet and bandlet and adaptive dictionaries are trained KSVD and Method of Directions (MOD) algorithms. The over-complete dictionary is generated by combining multiple standard transforms, including curvelet transform, ridgelet transform and discrete cosine transform. The over-complete predetermined dictionary such as wavelet-based dictionary, a shape-based and a region-based dictionary are used to represent the small image patch sparsely [6]. Currently, the most used transforms for execution are the Discrete Cosine Transform (DCT) and Discrete Wavelet Transforms (DWT). An important reason for the attractiveness of both these transforms is the feasibility of their fast implementation. Wavelet and contourlet transforms are broadly used in image processing applications. However, the difference between wavelet and contourlet is that wavelet transform has square compact support, whereas contourlet has directional rectangular compact support. By applying directional filter banks, contourlet is suitable for analyzing image contours.

In this paper we proposed the dictionaries based on discrete cosine transform and discrete wavelet transform based on difference of Gaussian function (DOG) and also apply optimization algorithms like basis pursuit (BP) and orthogonal basis pursuit (OMP) to measure the performance of our proposed dictionaries, after this we trained the dictionary by applying the KSVD algorithm and measure the performance using root mean square error. We first fix the dictionary and learn sparse coefficient based on KSVD algorithm. The DTT dictionary is introduced as a comparison with the proposed DWT and DCT dictionary, ether the dictionary based on DTT is perform well or not. Our proposed approach is to optimize the fixed or predetermined dictionaries using KSVD algorithm instead of using random based dictionary and second to optimize the objective function using basis pursuit as well as OMP for image in-painting application. The results shows that our proposed dictionaries are perform well by using BP as compared to the OMP dictionary.

2. RELATED WORK

Researchers, covering a several range of perspectives, have supported the use of over-complete signal representations. The dictionary D in the sparse representations can either be chosen as a pre-specified set of functions, such as Wavelets [8], Curvelets [9], or designed by adapting its content to suitable a given set of signal examples. A pre-specified dictionary is highly structured and leads to fast numerical implementations, with the drawback of rigidity to adjust the representation to the data.

The second approach is based on machine learning techniques to conclude the dictionary from a set of examples. It is a two-step procedure, in first step the starts with some initial dictionary and finds sparse approximations of the set of training signals while keeping the dictionary fixed. In the second step, the sparse coefficients are fixed while dictionary is optimized. The dictionary learning approach produce suitable results but it has more computational complexity than the pre-specified dictionaries and generate unstructured dictionary. Applications for over-complete representations have included multiscale Gabor functions [2], systems defined by algebraic codes [10], combinations of wavelets and sinusoids [11], collections of windowed cosines with a range of different widths and locations [12], multiscale windowed ridgelets [13], systems generated at random [14], and combinations of wavelets and line like producing elements [15]. A more formal approach convexifies (N_0) by replacing the l^0 -norm with an l^1-norm [16]

$$(N_0): b\|b_0\| \; subject\ to\ Y = DB \qquad (1)$$

An attractive approach to express over-complete signal representation is called BP [4]. Sometimes this approach provides excellent sparse solution. The important point is that the convex relaxation (BP) and greedy techniques are applicable to the sparse solution instead of using direct solution of(N_0). BP is a principle for decomposing a signal into an "optimal" superposition of dictionary elements, where optimal means having the smallest l^0-norm of coefficients among all such decompositions.

We provide examples exhibiting several advantages over MP, OMP comprising better sparsity and super-resolution. BP has interesting relations to ideas in areas as various as ill-posed problems, abstract harmonic analysis, total variation de-noising, and multiscale edge de-noising. BP in highly over-complete dictionaries leads to large-scale optimization problems [4].

3. MATERIAL AND PROPOSED METHODS

We proposed three dictionaries based on discrete cosine basis function (DCT) and discrete wavelet basis (DWT) .We proposed the difference of Gaussian function (DOG) in two dimensional with some scaling and dilation parameters.

A. Discrete Cosine Transform

In particular, a DCT is a Fourier-related transform similar to the discrete Fourier transform (DFT), but using only real numbers. DCTs are equivalent to DFTs of roughly twice the length, operating on real data with even symmetry. The most common variant of DCT is the type-II DCT.

B. Discrete Wavelet Transform

In our proposed method, we used the mother wavelet called difference of Gaussian (DOG) wavelet as a basis function to implement the discrete wavelet transforms and this mother wavelet is represented mathematically in following equations.

$$y_1(t) = 1/\sigma_1\sqrt{2\pi}e^{((t-)^2/2\sigma_1)} \quad (2)$$

$$y_2(t) = 1/\sigma_2\sqrt{2\pi}e^{((t-u)^2/2\sigma_2)} \quad (3)$$

The DOG function can be written as

$$F(t, u, \sigma_1, \sigma_2) = y_1(t) * y_2(t) \quad (4)$$

C. Discrete Tchebichef Transform (DTT)

The alternative of Discrete Cosine transform (DCT) is the Discrete Tchebichef Transform (DTT) for the image compression algorithms like JPEG. The DTT has the very similar properties to DCT and the only difference is that it has lower computation time and higher energy compression based on set of recurrence relation. The DTT is resulting from the very popular discrete class is called Tchebichef polynomials. It is new version of the orthonormal transform. Tchebichef polynomials has ration of applications in compression and image analysis [17]. DTT based on Tchebichef moments is produced the basis matrix as the DCT used the basis matrix as the trigonometric functions [18]. For image of size$N \times N$. The forward discrete Tchebichef transform can be written in the following form.

$$F_{n_1n_2} = \sum_{v=0}^{M-1}\sum_{u=0}^{M-1} f_{n_1}(v)\, f_{n_2}(u) f(v,u) \quad (5)$$

$$F(v,u) = \sum_{v=0}^{M-1}\sum_{u=0}^{M-1} F_{n_1n_2} f_{n_1}(v)\, f_{n_2}(u) \quad (6)$$

Where $n_1, n_2, v, u = 0, \dots \dots \dots, M-1$, parameters are used in forward as well as in reverse DTT function.

Now we defined the basis function of DTT. $f_{n_1}(v)$ is the vth order of the Tchebichef moments. These can be defined using the following function over the discrete range $[0, M]$.

$$f_{n_1}(v) = n_1! \sum_{t=0}^{n_1} -1^{n_1-1} \binom{M-1-t}{n_1-t} \binom{n_1+t}{n_1} \binom{v}{t} \quad (7)$$

$$F_{n_1n_2} = \sum_{v=0}^{M-1} f_{n_1}(v) \sum_{u=0}^{M-1} f_{n_2}(u)\, f(v,u) \quad (8)$$

$$F_{n_1n_2} = \sum_{v=0}^{M-1} f_{n_1}(v) f_{n_2}(v) \quad (9)$$

This is the final expression as shown in the equation (9) and this expression is mathematically equal to the expression of DCT.

D. K-SVD Algorithm

A generalization of the K-means algorithm for dictionary learning called the K-SVD algorithm has been proposed by Aharon [6]. The sparse approximation step is updated based on the orthogonal basis pursuit (BP) and update each column of the dictionary using singular value decomposition, in this way the residual and error can be minimized. Each patch of the signal has different weights and can be represented in the dictionary with multiple atoms. The only drawback of this algorithm is convergence and it may not produce good accuracy for the large signals or image patches. However, KSVD algorithm shows good performance for image de-noising applications. The basic steps of the algorithm can be optimized as shown in the following equations.

Given fixed sparse matrix B, The dictionary D can be updated by solving the following problem:

$$min_D\{\|Y - DB\|_F^2\}\ subject\ to\ \|b_i\|_0 \le N_0 \quad (10)$$

In sparse coding stage, there exist two assumptions. First, the D is fixed and the sparse coefficients are updated using the objective function as shown in the equation (10). The second assumption you can fixed the sparse coefficients and dictionary can be updated using any optimization algorithm like KSVD. The objective function can be solved using the SVD decomposition:

$$\begin{aligned}\|Y - DB\|_F^2 &= \left\| Y - \sum_{k=1}^{j} c_k b_T^k \right\|_F^2 \\ &= \left\| \left(Y - \sum_{k \ne j} c_k b_T^k \right) - c_k b_T^k \right\|_F^2 \\ &= \|E_k - c_k B_T^k\|_F^2\end{aligned} \quad (11)$$

Where ${b_j}^T$ rows of sparse matrix are B, c_k denotes the atom of the dictionary D and the E_k stands for the residual matrix. After the singular value decomposition (SVD) decomposition of the matrix E_k, both the atom c_k and ${b_j}^T$ can be updated. This SVD is time consuming process. We introduced the fixed dictionaries called DCT and DWT instead of using random to improve the accuracy and then apply the KSVD algorithm to further update the sparse coding stage adaptively.

The size of dictionary is 64×300,100×300, 144×200 and 256×300 and size of each atom in the dictionary is varied from 8 to 16. The examples are the patches of small image to train the dictionary using KSVD algorithm using zero-mean and unit norm dictionary atoms. The DWT and DCT are applied on the extracted patches with the selection of highest energy. The sparse model is solved using greedy algorithm i.e., BP and OMP and selects the best sparse coefficients from the sparse model. The overall execution steps are shown in the Table 1.

Algorithm	Sparse Coding based on fixed dictionary
Initialization :	Dimension of data
W	Size of patch
N	Dimension of data to be spared
M	Number of dictionary atoms stored
S	Sparsity factor
	Start:
1	Patch extraction from image $f = m \times m$
2	Extract sequences of patches $Y = \sum_{j=1}^{m} Y_j$
3	Choose zero-mean and unit norm atoms $Y = zeromean(Y) - norm(Y)$
	Select patch of highest energy Y
5	Use DWT and DCT dictionary to store the image patch
6	Apply Sparse model $\min_D \{\lVert Y - DB \rVert_F^2\}$ $subject\ to, \lVert b_i \rVert_0 \le N_0$
7	Solve Sparse Model using greedy algorithms like BP and OMP
8	End

Table 1. The sparse coding based on fixed dictionary

4. RESULTS

In result section, we proposed dictionaries based on DCT, DWT and DTT transform bases functions in order to compute the image in-painting problem using sparse representation. The proposed dictionaries are shown in Fig. (1,2,3). By visual inspection, the dictionary based on DWT bases function produced more accuracy as compared to other by capturing the corner as well as points from raw image patches.

The proposed dictionaries using KSVD algorithm in order to make adaptive of these fixed dictionaries and also optimized using optimization algorithms based on basis pursuit and orthogonal matching pursuit. The root mean square error is shown in the Fig. (4-7) based on DCT, DTT and DWT dictionaries. The dictionary based on DCT optimized using BP produced less error as compared to optimized using OMP algorithm as shown in the Fig. (4, 5) and also we observed that the size of the dictionary is an important factor for measuring accuracy. The 16×16 size dictionary produced less error as compared to other dictionaries having size 8×8, 10×10 and 12x12.

The convergence rate of size 8×8 and 10×10 is much better as compared to all proposed dictionaries. The dictionary based on DCT optimized using OMP as shown in the Fig.16 and size of the dictionary 8×8, 10×10 and 12×12 produce less error as compared to the dictionary size greater than 12.

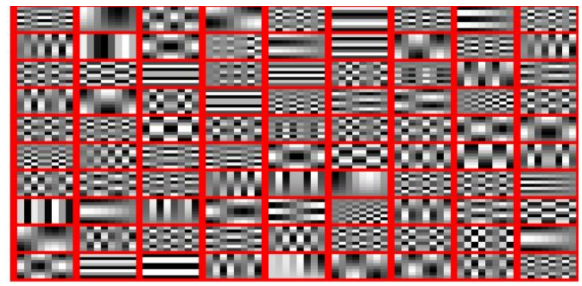

Fig. 1. Dictionary based on DCT (8×8 atom size)

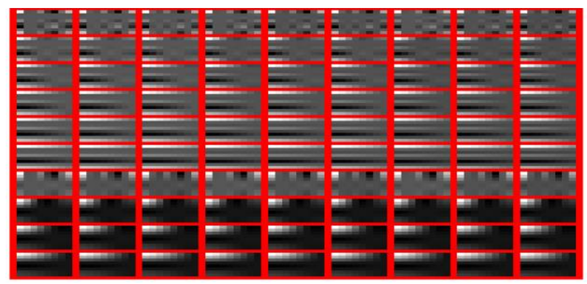

Fig. 2. Dictionary based on DWT basis function (8×8 atom size)

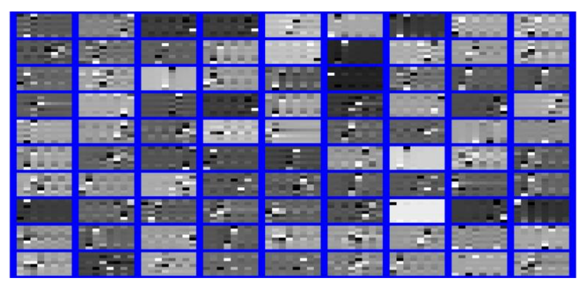

Fig. 3. Discrete Tchebichef dictionary has size (8x8)

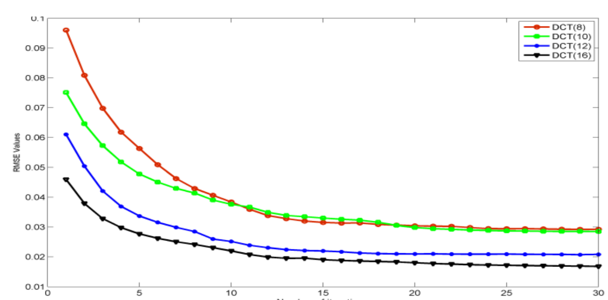

Fig. 4. Root mean square error (RMSE) of dictionary using DCT basis function based on Basis Pursuit (BP)

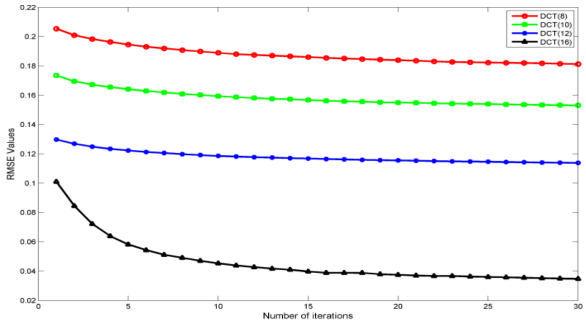

Fig. 5. Root mean square error (RMSE) of dictionary using DCT basis function based on Orthogonal Matching Pursuit (OMP)

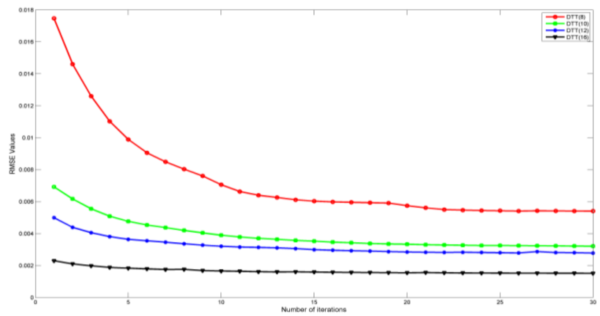

Fig. 6. Root mean square error (RMSE) of dictionary using DTT basis function based on Bsis Pursuit (BP)

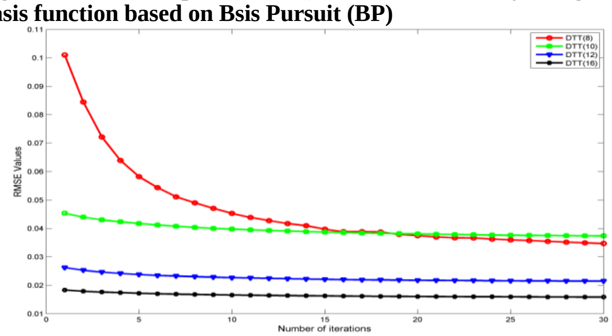

Fig. 7. Root mean square error (RMSE) of dictionary using DTT basis function based on Orthogonal Matching Pursuit (OMP)

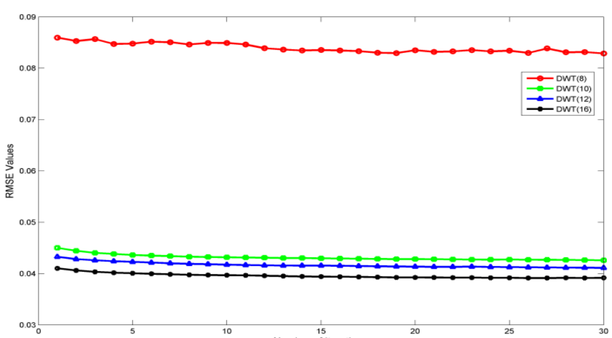

Fig. 8. Root mean square error (RMSE) of dictionary using DWT basis function based on Bsis Pursuit (BP)

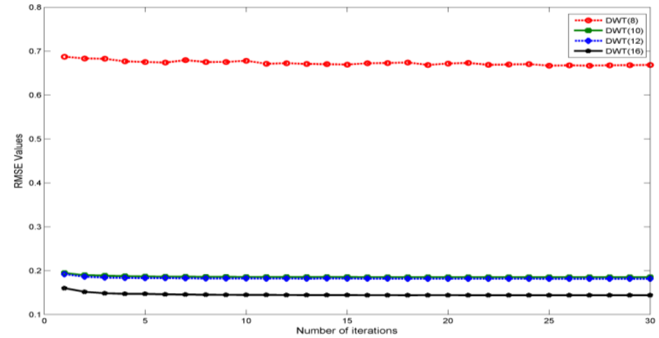

Fig. 9. Root mean square error (RMSE) of dictionary using DWT basis function based on Orthogonal Matching Pursuit (OMP)

4.1 Application to image in-painting

The experiments were done using the dictionaries based on basis functions (DCT, DWT and DTT). The three images were used for training of the size 16x16 pixels. The patches of the images are used as in vector form and removed the DC point by apply mean command in Matlab. For each patch, before reconstruction, the mean value has been removed and added back after reconstruction. Thus, DC component was artificially added in reconstruction yielding better results than it has been a part of the basis. To prevent border effects, reconstruction has been done such that the adjacent patches overlapped in two rows and two columns. After reconstruction overlapping regions were averaged. For a random pattern of missing pixels each in-painting experiment has been repeated 10 times and the final performance measure has been obtained as an average. Salt and pepper noise generated random pattern of missing values and that is the easiest in-painting problem to solve. Every experiment has been tested for three types of images with added noise and reconstructs the images based on the proposed dictionaries. The dictionary stored the image patch as shown in the Fig.10.

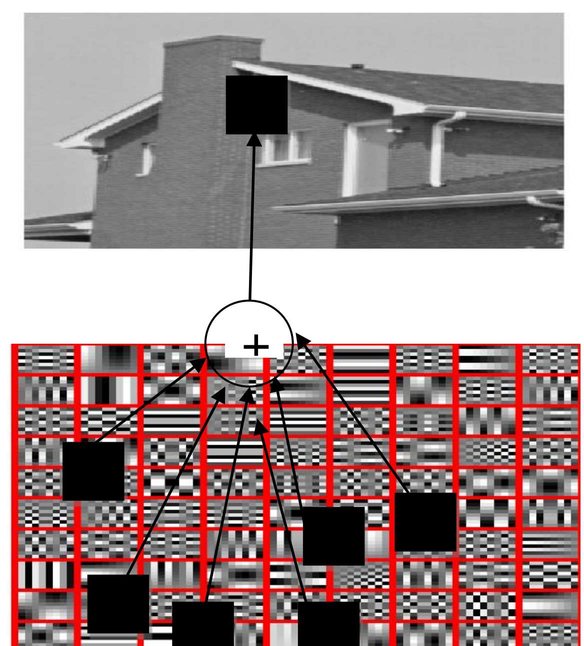

Fig. 10. The image patch stored in over-complete dictionary based on sparse representation

The missing patterns with a fixed dictionaries are difficult the find due to randomness of the patches selection. The size of the block and dictionary learned capability is more important and critical. The only possibility to reconstruct the images with block averaging the patches. The percentage of the known pixels is more than 80 % in our case. Table. 1 shows that the adaptive dictionary based on KSVD algorithm produced more accurate PSNR values as compared to the other dictionaries for image in-painting application for the dictionary size 8x8. In Fig. 15, the existing data set has been used for measuring the performance of the proposed dictionaries. KSVD based dictionary performs well as compared to the other dictionaries for in-painting application at the last column shown in Fig.15. The Fig.11 shows that the results of different PSNR values using the House image as an input for in-painting, the comparison was made on different fixed dictionaries i.e., DCT, DTT, DWT and adaptive dictionary using KSVD algorithm. From the Fig.12, the PNSR values clearly showed that the KSVD performed well with comparisons of fixed dictionaries. Similarly the results presented in Fig.12 using Lena Image and the results are expected similar as compared to the house image .The Fig. 13 shows the values of the PSNR in dB for Barbra image and again KSVD performs well as compared others. The Fig.14 handles the results by averaging the all three data set images values based on different dictionaries with different number of elements

Table. 1. PSNR values for image in-painting using Over-complete dictionaries

Image	DCT (8x8) PSNR	DTT (8x8) PSNR	DWT (8x8) PSNR	KSVD (8x8) PSNR
House Image	29.0321	29.8901	32.1047	**32.7102**
Lena Image	30.2501	30.7050	31.7952	**31.9616**
Barbra Image	27.5595	28.0191	29.3413	**29.71**

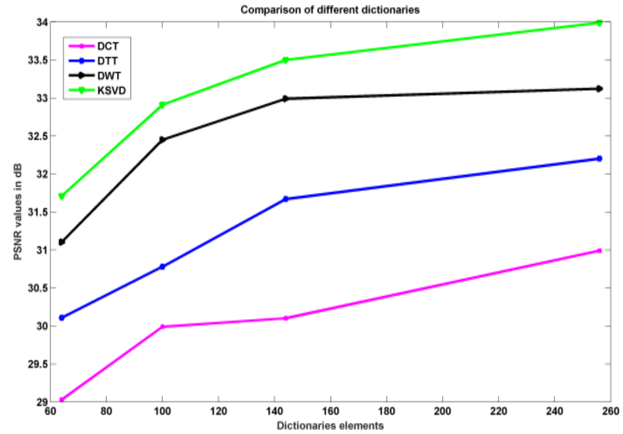

Fig. 11. Comparison of PSNR values in dB using different dictionaries based on House image with different dictionary elements (64,100,144,256)

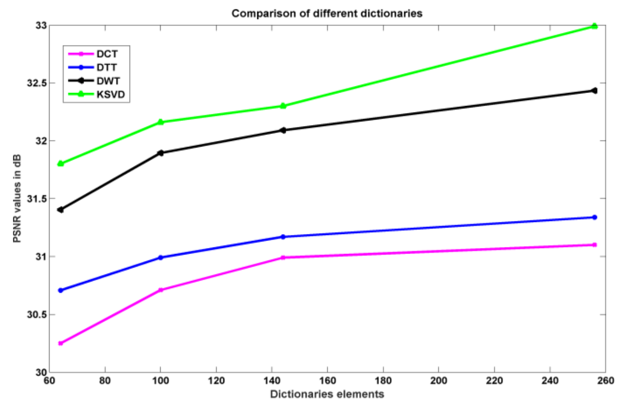

Fig. 12. Comparison of PSNR values in dB using different dictionaries based on Lena image with different dictionary elements (64,100,144,256).

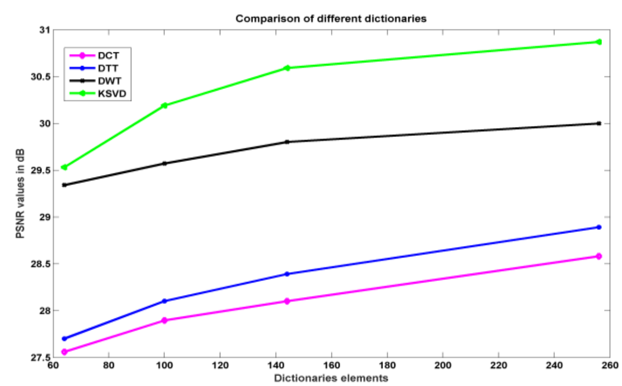

Fig. 13. Comparison of PSNR values in dB using different dictionaries based on Barbra image with different dictionary elements (64,100,144,256)

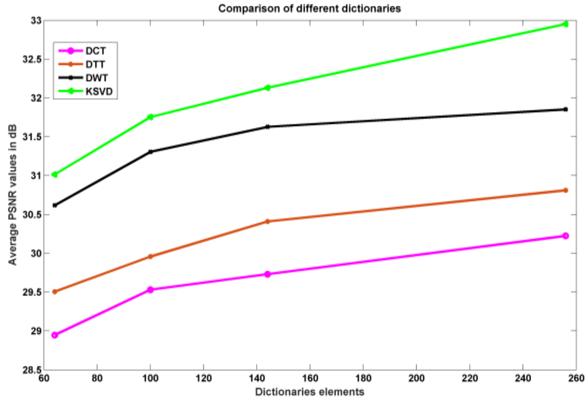

Fig. 14. Comparison of PSNR values in dB using different dictionaries based on House, Lena and Barbra with different dictionary elements (64,100,144,256)

CONCLUSION

In this paper, we proposed two dictionaries and further evaluated these dictionaries using optimization techniques called basis pursuit and orthogonal matching pursuit and results showed that the dictionary based on DWT produced accurate results as compared to the dictionary based on DCT. It is also concluded from the results that the BP optimize the sparse coefficients as compare to the OMP by calculating the average RMSE values. We trained our dictionaries using KSVD algorithm to further optimize the dictionary for approximating the sparse coefficients for sparse model. The fixed dictionaries are used to make the sparse model fast instead of using adaptive dictionary.

Original Images	Noisy images	In-painting using DCT dictionary	In-painting using DWT dictionary	In-painting using KSVD dictionary

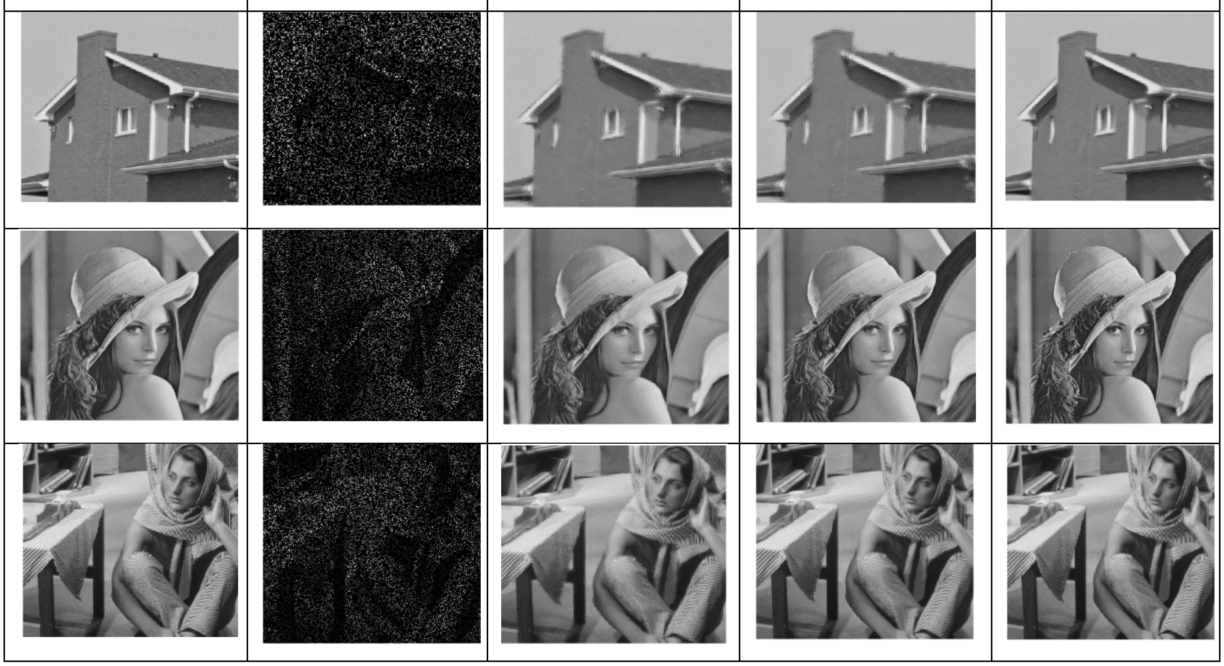

Fig. 15. Data set for the image in-painting based on the DCT, DWT and KSVD dictionaries have noisy patches.

In future, we can also use different predetermined dictionaries and also can investigate the optimization technique to further increase the performance of the sparse model.

REFERENCES

[1] Sadeghi, Mohammadreza, Massoud Babaie-Zadeh, and Christian Jutten. "Dictionary learning for sparse representation: A novel approach." Signal Processing Letters, IEEE 20.12 (2013): 1195-1198.

[2] S. Mallat and Z. Zhang. “Matching pursuits with time-frequency dictionaries,” IEEE Trans. Signal Processing., 41(12):3397–3415, 1993.

[3] J.A. Tropp. “Greed is good: Algorithmic results for sparse approximation,” IEEE Trans. Inform. Theory, 50(10):2231–2242, October 2004.

[4] Chen, Scott Shaobing, David L. Donoho, and Michael A. Saunders. "Atomic decomposition by basis pursuit." SIAM journal on scientific computing 20.1 (1998): 33-61.

[5] Tropp, Joel, and Anna C. Gilbert. "Signal recovery from random measurements via orthogonal matching pursuit." Information Theory, IEEE Transactions on 53.12 (2007): 4655-4666.

[6] Elad, Michael, and Michal Aharon. "Image denoising via sparse and redundant representations over learned dictionaries." Image Processing, IEEE Transactions on 15.12 (2006): 3736-3745.

[7] Elad, Michael, and Michal Aharon. "Image denoising via sparse and redundant representations over learned dictionaries." Image Processing, IEEE Transactions on 15.12 (2006): 3736-3745.
[8] W. T. Freeman and E. H. Adelson, "The design and use of steerable filters," IEEE Pattern Anal. Mach. Intell., vol. 13, no. 9, pp. 891–906, Sep. 1991.
[9] Donoho, David L., Michael Elad, and Vladimir N. Temlyakov. "Stable recovery of sparse over-completes representations in the presence of noise," Information Theory, IEEE Transactions on 52.1 (2006): 6-18.
[10] T. Strohmer and R. Heath Jr, "Grassmannian frames with applications to coding and communications," Appl. Comp. Harm. Anal., vol. 14, pp. 257–275, May 2003.
[11] Candès, Emmanuel J., and David L. Donoho. "New tight frames of curvelets and optimal representations of objects with piecewise C2 singularities," Communications on pure and applied mathematics 57.2 (2004): 219-266.
[12] R. Coifman, Y. Meyer, and V.Wickerhauser, "Adapted wave form analysis,wavelet-packets and applications," in Proc. 2nd Int. Conf. Industrial and Applied Mathematics (ICIAM 91), Philadelphia, PA, 1992, pp. 41–50.
[13] J.-L. Starck, E. Candès, and D. L. Donoho, "The curvelet transform for image denoising," IEEE Trans. Image Process., vol. 11, no. 6, pp. 131–141, Jun. 2002.
[14] Donoho, David L., and Michael Elad. "Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ1 minimization." Proceedings of the National Academy of Sciences 100.5 (2003): 2197-220.
[15] X. Huo, "Sparse Image Representation via Combined Transforms. Stanford," CA: Stanford Univ., Nov., 1999.
[16] E. J. Candes and D. L. Donoho, "Recovering edges in ill-posed inverse problems: Optimality of curvelet frames," Ann. Statist., vol. 30, no. 3, pp. 784–842, Jun. 2002.
[17] K.Nakagaki and R.Mukundan, "A Fast 4x4 Forward Discrete Tchebichef Transform Algorithm," IEEE Signal Processing Letters, vol.14, no.10, October 2007.
[18] R.Mukundan, "Image Analysis by Tchebichef Moments", IEEE Trans. on Image Proc., vol.10, no.9, pp.1357-1364, 2001.