# APPLICATION OF NEW VARIATIONAL METHOD USING HAMILTONIAN FOR NONLINEAR OSCILLATORS WITH DISCONTINUITIES

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*ABSTRACT**:** In this paper, we used Hamiltonian for nonlinear oscillators with discontinuities. The maximal relative error for the frequency obtained by new variational method compared with the exact solution indicates the remarkable precision of this method. Some examples are given to illustrate the effectiveness and convenience of the method.*

Keywords: Nonlinear oscillator; Variational principle, Hamiltonian, Analytical approximate solution .

## 1. INTRODUCTION

There are many approaches for approximating solutions to nonlinear oscillatory systems. The most widely studied approximation methods are the perturbation methods [24]. The simplest and perhaps one of the most useful of these approximation methods is the Lindstedt–Poincare´ perturbation method, whereby the solution is analytically expanded in the power series of a small parameter [20]. To overcome this limitation, many new perturbative techniques have been developed. Modified Lindstedt–Poincare´ techniques [21–23], the homotopy perturbation method [24–30] or linear delta expansion [31–33] are only some examples of them. A recent detailed review of asymptotic methods for strongly nonlinear oscillators can be found in [16]. The harmonic balance method is another procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation [17-20,34–42]. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required. In this research, we used new variational method using Hamiltonian [43] for nonlinear oscillators with discontinuities. We observe from the results that this method is very simple, easy to apply, and gives a very good accuracy even with the first-order approximation and simplest trial functions. Comparison made with other known results show that te method provides a mathematical tool to the determination of limit cycles of more complex nonlinear oscillators.

Suppose nonlinear oscillator

$$u'' + f(u) = 0, \tag{1}$$

with initial conditions

$$u(0) = A \text{ and } u'(0) = 0.$$

It is easy to establish a variational principle for equation (1), which reads [1]

$$J(u) = \int_0^{T/4} \left\{ \frac{1}{2} u'^2 - F(u) \right\} dt, \tag{2}$$

where $T$ is period of the oscillator, $\frac{\partial F}{\partial u} = f(u).$

In our previous study [1], variational approach to nonlinear oscillators was suggested, where the trial-function is chosen as

$$u = A\cos \mu t, \tag{3}$$

where $\mu$ is frequency. Substituting Eq. (3) into Eq. (2) results in

$$J = \int_0^{T/4} \left\{ \frac{1}{2} A^2 \mu^2 \sin^2 \mu t - F(A\cos \mu t) \right\} dt. \tag{4}$$

According to Ref. [1], the frequency–amplitude relationship can be obtained from the following equation

$$\frac{\partial J}{\partial A} = 0. \tag{5}$$

Explanation of Eq. (5) was given in Ref. [2]. This variational method for nonlinear oscillators have been used by many authors [3–6]. In this Letter we will develop a new variational method for nonlinear oscillators using Hamiltonian.

## 2. Hamiltonian

**I**n the functional (2), $\frac{1}{2}u'^2$ is kinetic energy and *F (u)* potential energy, so the functional (2) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads

$$H = \frac{1}{2}u'^2 + F(u) = \text{constant} = H_0, \tag{6}$$

or

$$\frac{1}{2}u'^2 + F(u) - H_0 = 0. \tag{7}$$

Equation (6) implies that the total energy keeps unchanged during the oscillation. In our previous work [7], $u = A\cos\omega t$ was used as a trial function and it was substituted to Eq. (7) to obtain a residual

$$R(t) = \frac{1}{2} A^2 \mu^2 \sin^2 \mu t - F(A\cos \mu t) - H_0. \tag{8}$$

Locating at some a special point, i.e., $\omega t = \pi/4$, and setting $R(t = \pi/4\omega) = 0$, we can obtain an approximate frequency–amplitude relationship of the studied nonlinear oscillator. Such treatment is much simple and has been widely used by engineers [8–15]. The accuracy of such location method, however, strongly depends upon the chosen location *point*. To overcome the shortcoming of the energy balance method, in this Letter we suggest a new approach based on Hamiltonian.

From Eq. (6), we have

$$\frac{\partial H}{\partial A} = 0. \qquad (9)$$

Introducing a new function, $\overset{\circ}{H}(u)$ defined as

$$\overset{\circ}{H}(u) = \int_0^{T/4} \left\{ \frac{1}{2} u'^2 + F(u) \right\} dt = \frac{1}{4} TH. \qquad (10)$$

It is obvious that

$$\frac{\partial \overset{\circ}{H}}{\partial T} = \frac{1}{4} H. \qquad (11)$$

Equation (9) is, then, equivalent to the following one:

$$\frac{\partial}{\partial A}\left( \frac{\partial \overset{\circ}{H}}{\partial T} \right) = 0, \qquad (12)$$

or

$$\frac{\partial}{\partial A}\left( \frac{\partial \overset{\circ}{H}}{\partial (1/\mu)} \right) = 0. \qquad (13)$$

From Eq. (13) we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator.

## 3. Applications

In order to assess the advantages and the accuracy of new method, we will consider the following examples:

**Example 1.** Consider a nonlinear oscillator with discontinuity in the form

$$u'' + \mu_0^2 u + £ u|u| = 0. \qquad (14)$$

With initial condition

$u(0) = A$ and $u'(0) = 0.$

It is easy to establish a variational principle for equation (14), which reads [1]

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2} u'^2 + \frac{\mu_0^2}{2} u^2 + \frac{1}{3} £ u^3 \right\} dt + \int_{T/4}^{T/2} \left\{ -\frac{1}{2} u'^2 + \frac{\mu_0^2}{2} u^2 - \frac{1}{3} £ u^3 \right\} dt, \qquad (15)$$

and $\overset{\circ}{H}(u)$ can be written in the form

$$\overset{\circ}{H}(u) = \int_0^{T/4} \left\{ \frac{1}{2} u'^2 + \frac{\mu_0^2}{2} u^2 + \frac{1}{3} £ u^3 \right\} dt + \int_{T/4}^{T/2} \left\{ \frac{1}{2} u'^2 + \frac{\mu_0^2}{2} u^2 - \frac{1}{3} £ u^3 \right\} dt. \qquad (16)$$

Assume that the solution can be expressed as $u = A \cos \omega t$, substitute in (16), we get

$$\overset{\circ}{H}(u) = \int_0^{T/4} \left\{ \frac{1}{2} A^2 \mu^2 \sin^2 \mu t + \frac{\mu_0^2}{2} A^2 \cos^2 \mu t + \frac{£}{3} A^3 \cos^3 \mu t \right\} dt + \int_{T/4}^{T/2} \left\{ \frac{1}{2} A^2 \mu^2 \sin^2 \mu t + \frac{\mu_0^2}{2} A^2 \cos^2 \mu t - \frac{£}{3} A^3 \cos^3 \mu t \right\} dt. \qquad (17)$$

$$\overset{\circ}{H}(u) = \frac{1}{\mu} \int_0^{®/2} \left\{ \frac{1}{2} A^2 \mu^2 \sin^2 t + \frac{\mu_0^2}{2} A^2 \cos^2 t + \frac{£}{3} A^3 \cos^3 t \right\} dt + \frac{1}{\mu} \int_{®/2}^{®} \left\{ \frac{1}{2} A^2 \mu^2 \sin^2 t + \frac{\mu_0^2}{2} A^2 \cos^2 t - \frac{£}{3} A^3 \cos^3 t \right\} dt. \qquad (18)$$

$$= \frac{1}{\mu}\left( \frac{16£ A + 9®\mu^2 + 9®}{36} \right). \qquad (19)$$

Now,

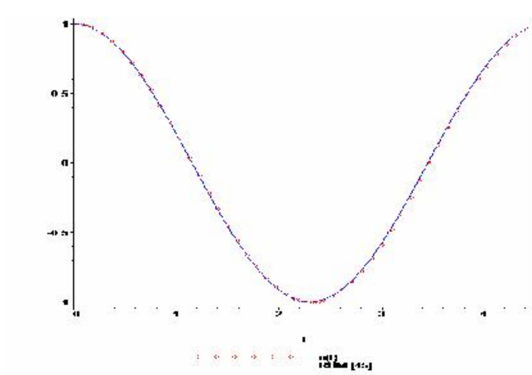
$$\left( \frac{\partial \overset{\circ}{H}(u)}{\partial (1/\mu)} \right) = -\frac{A^2 ®}{2} \mu^2 + \frac{A^2}{36}(16£ A + 9®\mu^2 + 9®\mu_0^2), \qquad (20)$$

and

$$\frac{\partial}{\partial A}\left( \frac{\partial \overset{\circ}{H}(u)}{\partial (1/\mu)} \right) = \frac{A}{3}\mu^2 \left( -3® + \frac{8£ A}{\mu^2} + \frac{3®\mu_0^2}{\mu^2} \right) = 0. \qquad (21)$$

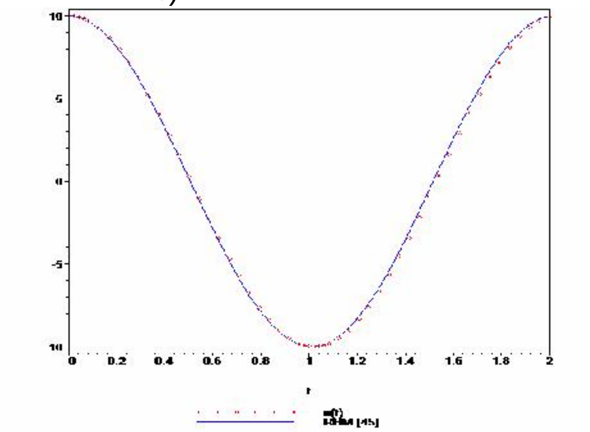
We get the following frequency-amplitude relation

$$\mu = \sqrt{\mu_0^2 + \frac{8}{3}\frac{£ A}{®}}. \qquad (22)$$

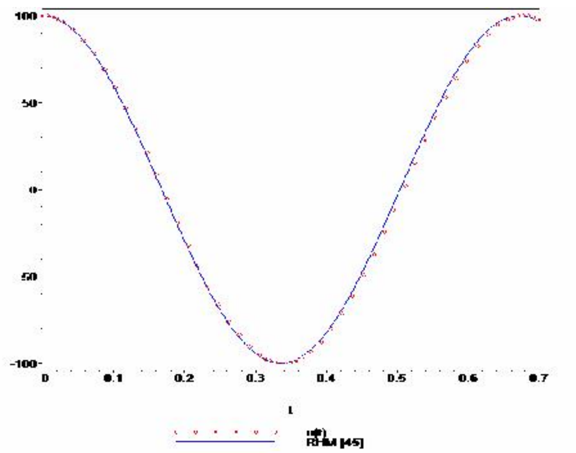


a) $\mu_0 = 1, £ = 1, A = 1$

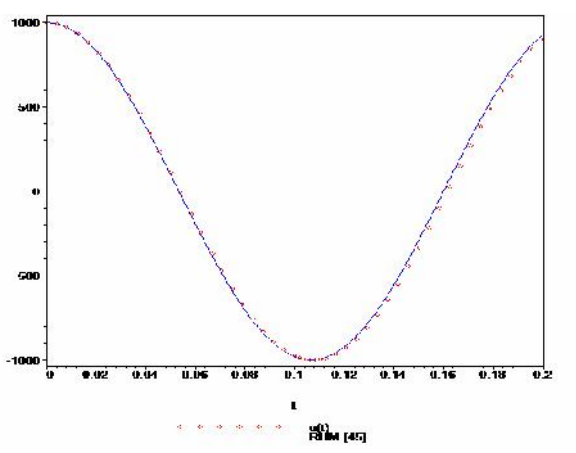
b)




c) $\mu_0 = 1, £ = 1, A = 10$




d) $\mu_0 = 1, £ = 1, A = 100$




e) $\mu_0 = 1, £ = 1, A = 1000$

Fig. 1

Fig.1 Comparison of the approximate solution $u = A\cos\mu t,$ where $\mu$ is defined by equation (22). Dashed line approximate solution; continuous line: Ren-He method (RHM, [45])

Same formula has been derived by using homotopy perturbation method [44]. In case $\mu_0^2 = 0, £ = 1,$ in equation (14) becomes $u'' + u|u| = 0,$ which has exact frequency $\mu_{ext} = 0.9147\sqrt{A},$ see [46], while our approximate frequency reads $\mu = 0.9310\sqrt{A},$ the accuracy is 1.78%, which can be considered an acceptable value if the simplicity of the method is taken into account.

**Example 2.** Consider the following nonlinear oscillator in the form

$$u'' + \quad u^3 + £u|u| = 0. \qquad (23)$$

with initial condition $u(0) = A$ and $u'(0) = 0.$

It is easy to establish a variational principle for equation (23), which reads [1]

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2}u'^2 + \frac{}{4}u^4 + \frac{1}{3}£u^3 \right\} dt + \int_{T/4}^{T/2} \left\{ -\frac{1}{2}u'^2 + \frac{}{4}u^4 - \frac{1}{3}£u^3 \right\} dt, \qquad (24)$$

and $\overset{\circ}{H}(u)$ can be written in the form

$$\overset{\circ}{H}(u) = \int_0^{T/4} \left\{ \frac{1}{2}u'^2 + \frac{}{4}u^4 + \frac{1}{3}£u^3 \right\} dt + \int_{T/4}^{T/2} \left\{ \frac{1}{2}u'^2 + \frac{}{4}u^4 - \frac{1}{3}£u^3 \right\} dt. \qquad (25)$$

Assume that the solution can be expressed as $u = A\cos\omega t,$ substitute in (25), we get

$$\overset{\circ}{H}(u) = \int_0^{T/4} \left\{ \frac{1}{2}A^2\mu^2\sin^2\mu t + \frac{}{4}A^4\cos^4\mu t + \frac{£}{3}A^3\cos^3\mu t \right\} dt + \int_{T/4}^{T/2} \left\{ \frac{1}{2}A^2\mu^2\sin^2\mu t + \frac{}{4}A^4\cos^4\mu t - \frac{£}{3}A^3\cos^3\mu t \right\} dt. \qquad (26)$$

$$= \int_0^{®/2} \left\{ \frac{1}{2}A^2\mu^2\sin^2 t + \frac{}{4}A^4\cos^4 t + \frac{£}{3}A^3\cos^3 t \right\} dt + \int_{®/2}^{®} \left\{ \frac{1}{2}A^2\mu^2\sin^2 t + \frac{}{4}A^4\cos^4 t - \frac{£}{3}A^3\cos^3 t \right\} dt. \qquad (27)$$

$$= \frac{1}{\mu}\left( \frac{27\ \ ®A^2 + 128£A + 72\mu^2®}{288} \right). \qquad (28)$$

Now,

$$\left( \frac{\partial \overset{\circ}{H}(u)}{\partial(1/\mu)} \right) = -\frac{A^2®}{2}\mu^2 + \frac{A^2}{288}(27\ \ ®A^2 + 128£A + 72\mu^2®), \qquad (29)$$

and

$$\frac{\partial}{\partial A}\left(\frac{\partial \mathring{H}(u)}{\partial(1/\mu)}\right)= \tag{30}$$
$$\frac{A}{24}\mu^2\left(-12® + \frac{9\ \ ®A^2}{\mu^2}+\frac{32£A}{\mu^2}\right)=0.$$

We get the following frequency-amplitude relation

$$\mu=\sqrt{\frac{3\ \ A^2}{4®}+\frac{8£A}{3®}}. \tag{31}$$

Same result was obtained [19], by using different technique. In this case choose $\ \ = 0, £ = 1,$ we obtained the same result [46].

## 4. CONCLUSION

In this research, we used new variational method using Hamiltonian for nonlinear oscillators with discontinuities. We observe from the results that this method is very simple, easy to apply, and gives a very good accuracy even with the first-order approximation and simplest trial functions. The discontinuous function will not affect the effectiveness and convenience of the method and solutions are valid for the whole domain.

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