COMPUTATION OF THE EULER CHARACTERISTIC OF THE MILNOR FIBRE OF PLANE CURVE SINGULARITIES

Muhammad Ahsan Binyamin, Rabia, , Saba Naz, Sonia Tariq
Department of Mathematics, Government College University Faisalabad, Pakistan
E-mail: ahsanbanyamin@gmail.com, rabiasaeed850 ,saba.aziz252, soniatariq826 (@yahoo.com)

ABSTRACT: *In this article we give the implementation of two algorithms to compute the Euler characteristic of Milnor fibre of reduced plane curve singularities in computer algebra system SINGULAR.*

Keywords-: Euler characteristic of Milnor fibre, Hamburger-Noether expansion, Computer algebra system SINGULAR, Irreducible curve singularities.

1. INTRODUCTION

In this short note we want to compare two approaches to compute the Euler characteristic of the Milnor fibre of a plane curve singularity. From the computational point of view, one approach uses the resolution graph and the multiplicity sequence of the curve. The other approach computes the irreducible decomposition using the Hamburger-Noether expansion and computes the Euler characteristic in terms of the irreducible components. Examples show that the second approach is much faster (see section-3).

Let $(C,0)$ be the germ of plane curve defined by the equation $f=0$, where f is an analytic function of two complex variables. Then the following definitions and results can be found in [4,5,6] .

Definition 1.1. The curve C defines as a line $L:=C\cap S_{\varepsilon}^{3}$ in the sphere of radius ε around $0\in C^{2}$, the line does not depend on ε if ε is small enough. Then the map $f/|f| : S_{\varepsilon}^{3}/L \rightarrow S^{1}$ is a $C^{\infty}-$locally trivial fibration, the Milnor fibration. Any fibre F of this fibration is called the Milnor fibre of f.

Definition 1.2. Let $(C,0)$ be an irreducible curve singularity then its Milnor number is define as

$$\mu(C)=\dim\frac{\square\; x,y}{\langle \partial f/\partial x, \partial f/\partial y\rangle}.$$

Now we are giving the formulae to compute the Euler characteristic of Milnor fibre

Proposition 1.3. Let $f=f_1\cdots f_r$ be the factorization in to irreducible factor, such that f_i/f_j, $1\le i,j\le r, i\neq j$, are as power series not unit. Let $(C,0)$ be the germ of plane curve defined by the local equation $f=0$ and $(C_i,0), i=1\cdots r$, be its branches defined by $f_i=0$, if F is the Milnor fibre of f then Euler characteristic of F can be computed as follows

$$\chi(F) = -2\sum_{i\le j, j\le r}\left(C_i, C_j\right)+\sum_{i=1}^{r}\left(1-\mu(C_i)\right)$$

Where $\left(C_i, C_j\right)$ is the intersection multiplicity of C_i and C_j at the origin and $\mu(C_i)$ is the Milnor number of C_i at the origin.

Proposition 1.4. Let G_f be the embedded resolution graph of plane curve singularity define by f. Let W is the set of all vertices corresponding to exceptional divisor and A is the set of vertices corresponding to the strict transform and $V=W\coprod A$. For each $w\in W$ let $\delta_w = cardinality\,of\,V_w$, where V_w is the set of vertices $v\in V$ adjacent to $w\in W$ then $\chi(f)=\sum_{w}m_w(2-\delta_w)$. Where m_w is the multiplicity sequence of plane curve defined by f.

2. THE ALGORITHMS

In this section we present the algorithms to compute the Euler characteristic of the Milnor fibre of plane curve singularity f which can be implemented in SINGULAR [2,3]

Algorithm-1: (Eulerchrac)

Input: A polynomial f, (defines a reduced curve singularity).

Output: An integer E, the Euler characteristic of plane curve singularity.

- Factorize $f=f_1\cdots f_r$.
- Compute a list H , the Hamburger-Noether expansion for all f_i.
- Compute the matrix of intersection multiplicities corresponding to the irreducible branches of f.
- compute the Milnor number m_i of each branch f_i of f
- compute the Euler characteristic E of the Milnor fibre of plane curve singularity as described in Proposition 1.3.
- Return(E)

Algorithm-2: (resEuler)

Input: A polynomial f, (defines a reduced curve singularity).

Output: An integer E, the Euler characteristic of plane curve singularity.

- Compute resolution graph of reduced plane curve singularity f.
- Compute sequence of multiplicities of reduced plane curve singularity f.
- Compute the Euler characteristic E of the Milnor fibre of plane curve singular as described in Proposition 1.4.
- Return(E)
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3. TABLE AND TIMINGS

In this section we provide some examples and a table which gives a time comparison between the 1st approach and the 2nd approach. Timings are conducted by using Singular 3-1-3 on an Intel®T2400, dual core 1.83 GHz processor, 1 GB RAM under the Window 2007 operating system. We consider the following polynomials:

$$\begin{aligned}
&1.\ f_1=\left(\left(x^7-y^6\right)^4-x^{20}y^{10}+x^5\right).\left(\left(x^7-y^6\right)^3-x^{24}+y^{25}\right)\\
&\qquad .\left(\left(x^7-y^5\right)^3-x^{24}-y^{25}\right),\\
&2.\ f_2=\left(x^{14}+x^5y^5+y^{10}+y^{15}\right).\left(x^{16}+x^{18}+y^{20}+y^{23}\right),\\
&3.\ f_3=\left(x^{14}+x^{10}y^4-y^{12}+y^{19}\right).\left(x^{12}+x^{15}-y^{14}+y^{15}\right).\\
&\qquad \left(x^{14}+y^{18}+y^{21}\right).\left(\left(x^6-y^5\right)^4-x^{26}y^{12}+x^{28}+y^{30}\right),\\
&4.\ f_4=\left(x^8+2y^{14}\right).\left(x^{10}+5y^{10}\right).\left(y^2-x^3\right).\left(x^2-y^3\right),\\
&5.\ f_5=\left(x^{18}+x^{29}+y^{24}\right).\left(x^{14}+y^{18}+y^{21}\right).\left(x^{15}-y^{10}-y^{19}\right)\\
&\qquad .\left(x^9-x^3y^3+y^{11}\right),\\
&6.\ f_6=\left(\left(x^7-y^6\right)^4-x^{20}y^{10}+x^{35}\right).\left(\left(x^7-y^6\right)^3-x^{24}+y^{25}\right),\\
&7.\ f_7=\left(x^{18}+x^{29}+y^{24}\right).\left(x^{14}+y^{18}+y^{21}\right).\left(x^9-x^3y^3+y^{11}\right),
\end{aligned} \tag{1}$$

Poly	$\left(1^{st}\ approach\right)$	$\left(2^{nd}\ approach\right)$
f_1	$2\sec$	$27\min$
f_2	$2\sec$	$40\min$
f_3	$3\sec$	$55\min$
f_4	$3\min$	$>1hr$
f_5	$3\sec$	$>1hr$
f_6	$2\sec$	$1\min$
f_7	$2\sec$	$4\min$

4. THEORETICAL COMPERISON

Now we compare the two formulae. In the 1st approach initially we make the factorization of $f=f_1\cdots f_r.$. After this we compute the Hamburger-Noether expansion of each f_i, then compute the intersection multiplicity of the branches appearing in the Hamburger-Noether expansion and also the Milnor number of each branch. In the second approach we compute the resolution graph and multiplicity sequences of f. For details see [1].

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