RAINBOW COLORING OF 3-TERM ARITHMETIC PROGRESSION

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Abstract: *In recent years Rainbow Ramsey Theory has been the focus of research to many combinatorists. Combinatorists have been working to find order out of randomly disordered sets. Three color classes were obtained but without proper order. If two numbers were chosen from two color classes then the third number was not from the third class, while it was from one of the chosen two classes. In this paper counting technique has been applied to find rainbow coloring. The existence of a rainbow AP(3) has been established even in the presence of a dominant color.Under new conditions, whenever two numbers are chosen from two different color classes, the third number is always from the third color class, so form AP(3).*

Keywords*:Arithmetic Progression, Color classes,order, density condition, dominant colors*.

INTRODUCTION

The branch of combinatorial Mathematics Known as Ramsey Theory concerns the existence of highly regular pattern in sufficiently large set of randomly selected objects like piles of stones, stars on the sky and piles of books and gathering of people.Pattern can be formed out of randomness in different ways.

For example, consider the numbers 1 to 10. If 1, 3 and 7 are colored red, 4, 8 and 10are colored green, and 2, 5, 6, 9 are colored blue, then a pattern of a 3-term AP is foundi.e. 2, 3, 4 this is a rainbow AP.

Ramsey Theory says that complete disorder is impossible. Somehow, no matter how complicated or random a set of events appears, a subset of definite pattern can be formed. Rainbow Ramsey Theory is widely used in Combinatorics particularly in well ordering theorem, graph theory for coloring of edges or vertices and number theory for placing numbers in different sets under certain conditions on sets, plane geometry, Logics as well as in other areas of mathematics.

The recent literature on the existence of rainbow AP(3) in 3-colorings and the restrictions on the density of each color as well as the cardinality of rainbow AP(3)'s is a motivation to investigate the situations in further depth.

In this paper the existence of a rainbow AP(3) under certain conditions and in the presence of a dominant color has been established.In this paper same conditions have been used to get rainbow 3-coloring with dominant colors.

In this paper, the existence of rainbow AP(3) in 3-colorings of N and [n] has been investigated. In particular, the result of Fon-Der Flaass and Axenovichhas has been generalized by finding a 3-coloring with a dominant color and yet having a rainbow AP(3). Three color classes have been used, i.e.R(red), B(blue) and G(green).

LITERATURE REVIEW

In [10], it was commented that ‘Complete disorder is impossible’. Somehow no matter how complicated or random set of events appear, we can find a subsetwhich has a definite pattern.

It has been shown in [13] that the equation x + y = z, has rainbowsolutions with x, y and z belonging to different color classes. Such solutions are calledrainbow solutions.

In the equation x+y = 2z for 3-coloring of [n] = 1, 2, 3,…,n, it has a rainbow solutionwith x, y and z belonging to different color classes and forming an AP(3) [16].

Van Der Waerden derived a result that for every choice of positive integer k and n,there exists a least N(k, n) = N such that for every partition of the set f1, 2, 3,...Nginto k classes, one of the classes contains an Arithmetic Progression with n terms [16].

It was shown that the largest size of one color class ensures the existence of monochromatic AP. So, to ensure the existence of rainbow AP(m), the size of all color classesshould be enlarged. Here m is the number of coloring i.e 3- coloring, 4-colouring etc. [14].

In [8] it has been proved that for equinumerous3-coloring of [n] there exists rainbow AP (3). It is very interesting to note that the minimal density for the color classes is $\frac{1}{6}$. Mahdian and Radoicic found rainbow solutions to the equation x + y = 2z in the3-coloring in the presence of a dominant color [6].

In [4] authors left the following theorem unsettled, “Every equinumerous k-coloring of [kn] contains a rainbow AP (k) iff k = 3”. It was suggested that for the existence of a rainbow AP(3) dominancy of one color should be reduced [9].

DEFINITIONS

Monochromatic Coloring: Monochromatic coloring of the elements of given set by using k-colors is the existenceof a subset whose all elements have same color. In term of an integer solution of a diophantine equation, it means all variables have the same color.

Example: Consider an equation x + y = 2z, here for x = 1, y = 1 we get z = 1, which gives thesame value for different variables of the same equation.

Rainbow Coloring: A Rainbow is the arch having colors of the spectrum formed in the sky oppositethe sun, when it is raining or when the sun shines on mist. In Mathematics, given a coloring of N, a set S a subset of N is called rainbow, if all elements of S are colored with different colors.

Example: Consider an equation x + y = 2z, here for x = 2 and y = 4, we get z = 3, so three variables have different values for the solution of the equation. So x, y and z belong to three different color classes. Thus 2, 3, 4 is a rainbow solution to the equation.

Arithmetic Progression: Arithmetic progression is the sequence of numbers in which the difference of two consecutive numbers remains the same. In short, it is written as AP. An m-term APmeans arithmetic progression of length m.

Example: 2, 5,8…is an AP, here difference between two consecutive numbers is same through-out the sequence.2, 5, 8, 11 is a 4-term AP.

Color Classes: Classes are actually partitions of a set into disjoint subsets. If the elements of each of these classes have a unique color and no two classes represent the same color. Then,these classes are called color classes.

Example: B, G, R and etc are the names assigned to color classes. Here B represents blue, Grepresents green and R represents red color class.

Cardinality of a Class: Cardinality of a class is the size of that color class.

Example:B = {1, 2, 3, 4} has cardinality of 4.

Density Condition: A condition which ensures certain minimum numbers in each class is called density condition.

Example: Density condition in rainbow coloring is greater than$\frac{1}{6}$.

RAINBOW FREE 3-COLORING

Theorem 1: There is a rainbow free 3-coloring c of N s.t for every n, $\min(|B|,|G|,|R|) = \frac{n+2}{6}$

Proof: Define a coloring on N,

c(j) = B: if j≡ 1(mod6)
= G: if j≡ 4(mod6)
= R: if j ≡ 0, 2, 3 and 5(mod6)

Here the numbers in the blue color class are of the form $6k_1$ +1, the numbers in a green color class are of the form $6k_2$ + 4 and the numbers in a red color class are of theform $6k_3$ + t, where t = 0, 2, 3, 5. The blue and the green colors cannot be addedbecause one is odd and the second is even, so sum will not give twice the mid valueof an AP(3). Sum of the blue class number with the red class number will also notgive twice of the green class number and the same is with the sum of green and thered class numbers. Therefore, sum of the two numbers from two different classes willnot give twice of the third one, so form no rainbow AP(3).

Size of the Smallest Color Class in a RainbowFree 3-coloring

Theorem 2: For every n ≥ 3, there is a rainbow free 3-coloring c of N, in whichsize of the smallest color class is $\frac{(n+4)}{6}$

Proof: For n ≡ 2(mod6) or n = 6k + 2, for an integer k.

Define coloring c on [n]:

c (j) = B: if j≤ 2k + 1 and j is odd
= G: if j≥ 4k + 2 and j is even
= R: otherwise

Since every blue number comes at most 2k + 1 and every green number occurs atleast 4k +2, a blue and a green cannot be the first and the second term of an AP(3),because a green number becomes twice of a blue number so, no place for the third term of anAP(3). Similarly, a blue and a green cannot become the second and thirdterm of an AP because again a blue B becomes the twice of a green G and no place forfirst term of an AP. Also, the blue numbers are odd and the green numbers are even,so cannot be the first and the third term of an AP. Hence, these distinct numbers of distinct classes do not form 3-term AP, which shows that c contains no rainbow AP(3). Now, the numbers in a blue color class are k+1, the numbers in a green colorclass are k+1, and the numbers in a red color class are 4k. Then,

$\min(|B|,|G|,|R|)$ = k + 1
$=\frac{(n-2)}{6}+1$
$=\frac{n+4}{6}$

This shows that the size of smallest color class is$\frac{n+4}{6}$.

Coloring with a Dominant Color

Proposition 1:There exists a 3-coloring which satisfies the following condition$\min(|B|,|G|,|R|)$ = r(n);

where r(n) =$\frac{(n+2)}{6}$, if n ≡2(mod6)
=$\frac{(n+4)}{6}$, otherwise

Such that it contains a rainbow AP(3).

Proof: Define a coloring c of N,

c(j) : = B if j≡1(mod6)
= G if j≡ 2(mod6)
= R if j≡ 0, 3, 4 and 5(mod6)

Here, in the above coloring a red color is the dominant. By using counting technique,the numbers in the blue color class are of the form 6k + 1, the numbers in the green color class are of the form 6k+2 and in the red color class are 6k+j, where j = 0, 3, 4, 5.6k +1 and 6k +3 are added to give twice of 6k +2. So 6k +1, 6k +2, 6k +3 form an AP(3), which is a rainbow AP(3), because all the three numbers belong to distinct color classes. Hence, by changing conditions on coloring classes, we can get a rainbow AP(3), even in the presence of a dominant color. Also, $\min(|B|,|G|,|R|) = \frac{n+2}{6}$. Now,

define a coloring for n = 6k + 2

c(j) : B = j≤ 2k + 1 j is odd
G = j ≤ 4k + 2 j is even
R = otherwise:

In this coloring the numbers in a blue color class are k + 1, the numbers in a green color class are 2k + 1 and the numbers in a red color class are 3k. Also, there can be found at least one AP(3) in this coloring.

$\min(|B|,|G|,|R|)$ = k + 1= $\frac{(n+4)}{6}$.

Existence of a Rainbow 3-Coloring of [n] witha dominant color

Proposition 2: There exists a rainbow 3-coloring of [n] with a dominant color.

Proof: Consider three colors B, G, R. If for some i, B color is taken at position 6k+i,where k = 0, 1, 2, 3 and G is taken at i + 6t + d, where d = 1, 2,3 then, R is alwaysfound at some 2(2i+d)+6(2k+t), these three colors will always form an AP(3), which is a rainbow AP(3). Define coloring c of [n],

c(j) = B: if j≡ 0(mod6); j = 6p; p is a positive integer
= G: if j≡1(mod6); j = 6q + 1; q is a positive integer
= R: if j ≡ 2, 3, 4, 5(mod6); j = 6t + m, j = 2,3, 4, 5 and t is a positive integer.

When n = 6k,

Numbers present in B class = k;
Numbers present in G class = k; and
Numbers present in R class = 4k:

$\min(|B|,|G|,|R|) = k = \frac{n}{6} = \frac{n+2}{6} - \frac{1}{3} < \frac{(n+2)}{6}$

When n =6k + 1,
Numbers present in a B class = k
Numbers present in a G class = k + 1 and
Numbers present in a R class = 4k
$\min(|B|, |G|, |R|) = k = \frac{n-1}{6}$
$= \frac{(n+2)}{6} - \frac{1}{2} < \frac{(n+2)}{6}$
When n = 6k + 3,
Numbers present in a B class = k
Numbers present in a G class = k + 1 and
Numbers present in a R class = 4k
$\min(|B|, |G|, |R|) = k = \frac{(n-3)}{6} = \frac{(n+2)}{6} - \frac{5}{6} < \frac{(n+2)}{6}$
Similarly, for n = 6k + 2, n = 6k + 4 and n = 6k + 5, the following condition holds with
$\min(|B|, |G|, |R|) < \frac{(n+2)}{6}$

Coloring without a Dominant Color

Proposition3: There exists a 3-coloring with no dominant color and satisfyingthe following condition
$\min(|B|, |G|, |R|) > r(n);$
where $r(n) = \frac{(n+2)}{6}$ if $n \not\equiv 2 \pmod 6$
$= \frac{(n+4)}{6}$ if $n \equiv 2 \pmod 6$
Such that it contains a rainbow AP(3).

Proof:Define the coloring of N,
c(j): = B if j≡ 1, 2(mod6)
= G if j≡ 3, 4(mod6)
= R if j≡ 0, 5(mod6)
This coloring is for the case of n ≢ 2(mod6)

Case-1:
When n ≢ 2(mod6), then the followingconditions arise,
a. n ≡0(mod6) or n = $6k_1$
b. n ≡ 1(mod6) or n = $6k_2 + 1$
c. n ≡ 3(mod6) or n = $6k_3 + 3$
d. n ≡ 4(mod6) or n = $6k_4 + 4$
e. n ≡ 5(mod6) or n = $6k_5 + 5$

a. For n ≡ 0(mod6) or n = $6k_1$:

Blue color class:
Here, for a blue color class condition is either j = 6s+1 or 6s+2, where s = 0, 1, 2, 3.
In the given condition the numbers in a blue color class are k + k = 2k.

Green color class:
Here the conditions for a green color class are j = 6t+3 or 6t+4, where t = 0,1, 2, 3.
It can be shown that the numbers in a green color class = k + k = 2k.

Red color class:
For a red color class the defined conditions are j = 6f or 6f + 5, where f = 0, 1, 2, 3.
So, the numbers in a red color class are k + k = 2k.

Condition on classes:
Writing partitioning of the natural numbers into three color classes in the form:
c(j) = B: totalnumbers = 2k
= G: total numbers = 2k
= R: total numbers = 2k
Then,
$\min(|B|, |G|, |R|) = 2k$
$= 2(\frac{n}{6}) = \frac{n}{3} > \frac{(n+2)}{6}$
$\min(|B|, |G|, |R|) > \frac{(n+2)}{6}$

Condition for finding rainbow AP(3):
As the condition of having minimum numbers in any color class greater than $\frac{(n+2)}{6}$ is satisfied, the next target is to ensure the existence of an AP(3). The numbers in a blue color class are of the form, i = $6k_1$ + 1 or $6k_3$ + 2. The numbers in a green color class are of the form, i = $6k_3$ + 3 or $6k_4$ + 4, and the numbers in a red color class are of the form, i = $6k_5$ or $6k_6$ + 5. Here $6k_5$, $6k_3$ + 2, $6k_4$ + 4 form an AP(3), where all the three numbers belong to different classes, so form a rainbow AP(3).

b.For n ≡ 1(mod6) or n = $6k_2$ + 1:

Blue color class:
In a blue color class the condition is either j = 6s + 1 or 6s + 2, where s = 0, 1, 2, 3.
The numbers for j = 6s+2 are k. The total numbers present in a blue color class are (k + 1) + k = 2k + 1.

Green color class:
Here, the condition for a green color class is j = 6t + 3 or 6t + 4, where t = 0, 1, 2, 3, then the numbers in a green color class are k + k = 2k.

Red color class:
The defined conditions are j = 6f or 6f + 5, where f = 0, 1, 2, 3. So, the total numbers are k + k = 2k.

Condition On classes:
Writing partitioning of the natural numbers into three color classes in the form:
c(i) = B: total numbers = 2k + 1
= G: total numbers = 2k
= R: total numbers = 2k
Then,
$\min(|B|, |G|, |R|) = 2k = 2(\frac{n-1}{6}) = \frac{n-1}{3} > \frac{n+2}{6}$
$\min(|B|, |G|, |R|) > \frac{(n+2)}{6}$

c. For n ≡ 3(mod6) or n = $6k_3$ + 3:

Blue color class:

Condition On classes:
c(j) = B: total numbers = 2k + 2
= G: total numbers = 2k + 1
= R: total numbers = 2k
Then, $\min(|B|, |G|, |R|) = 2k = 2(\frac{n-3}{6}) = \frac{n-3}{3} > \frac{n+2}{6}$
$\Rightarrow \min(|B|, |G|, |R|) > \frac{n+2}{6}$

d.For n ≡ 4(mod6) or n = $6k_4$ + 4:

Condition on classes:
c(j) = B: total numbers = 2k + 2
= G: total numbers = 2k + 2
= R: total numbers = 2k
Then,
$\min(|B|, |G|, |R|) = 2k = 2(\frac{n-4}{6}) = \frac{n-4}{3} > \frac{n+2}{6}$
$\Rightarrow \min(|B|, |G|, |R|) > \frac{n+2}{6}$

e. For n ≡ 5(mod6) or n = $6k_5$ + 5

Condition On classes:
c(j) = B: total numbers = 2k + 2
= G: total numbers = 2k + 2
= R: total numbers = 2k + 1

Then,

$\min(|B|, |G|, |R|) = 2k + 1$

$= 2(\frac{n-5}{6}) + 1$

$= \frac{n-2}{3} > \frac{n+2}{6}$

Hence, by all the five cases, the following condition is satisfied, i.e.

$\min(|B|, |G|, |R|) > \frac{n+2}{6}$

Case-2: (When there exists a dominant color)

When n ≡ 2(mod6) or n = 6k + 2,

Define coloring on N:

c(j) = B: if j≤ 2k + 3 and i is odd;

= G: if j≤ 4k and i is even;

= R: otherwise:

For a Blue color class:

it can be easily shown that for j≤ 2k + 3 the total numbers in the class are k + 2.

For Green color class:

The numbers in this class under the condition j≤ 4k are k + 2.

For Red color Class:

The total numbers in this class can be found by subtracting the numbers of other two classes from the total numbers i.e. 6k + 2 - (k + 2) - (k + 2) = 4k - 2.

Condition on color classes:

$\min(|B|, |G|, |R|) = k + 2 > \frac{n+4}{6}$

Therefore,

$\min(|B|, |G|, |R|) > \frac{n+4}{6}$

Existence of a rainbow AP(3):

Since all the blue class numbers are odd and the green class numbers are even, so they cannot be the first and the third terms of an AP(3). Whether a blue and a green number takes place the first and the second or the second and the third value of an AP(3). At the remaining place a number will come from a red color class. Hence, under the defined conditions there exists an AP(3).

Example:

For n ≡ 4(mod6). When n = 58 = 6*9 + 4

Then,

B(n) = {1, 2, 7, 8, 13, 14, 19, 20, 25, 26, 31, 32, 37, 38, 43, 44, 49, 50, 55, 56}

G(n) = {3, 4, 9, 10, 15, 16, 21, 22, 27, 28, 33, 34, 39, 40, 45, 46, 51, 52, 57, 58}

R(n) = {5, 6, 11, 12, 17, 18, 23, 24, 29, 30, 35, 36, 41, 42, 47, 48, 53, 54}

For (n) = $\frac{n+2}{6}$

$= \frac{58+2}{6} = 10$

Then,

$\min(|B|, |G|, |R|) = R(n) = 18$

$> 10 = \frac{n-4}{6} > \frac{n+2}{6}$

which shows that an AP (3) is present. For n ≡ 2(mod6) or n = 6k + 2, When n =56

B(56) = {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21}

G(56) = {36, 38, 40, 42, 44, 48, 50, 52, 54, 56}

R(56) = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 23, 24, 25, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55}

Here, $\min(|B|, |G|, |R|) = 11 > 10$ or $\frac{n+10}{6} > \frac{n+4}{6}$

Here, an AP(3) is 21, 36, 51.

Remarks:

In the previous proposition, it has been found that the number of APs in a rainbow coloring is > n where n is a number from the set of the natural numbers, which is partitioned into three color classes.

Existence of a rainbow 3-term AP:

Take any number $6k_1$ from a B color class and $6k_2+1$ from a G color class, then there exists a number $6k_3+2$ in R color class. So, $6k_1$, $6k_2+1$, $6k_3+2$ form AP which is a rainbow AP (3). The existence of a dominant color does not ensure the existence of a rainbow free coloring.

CONCLUSION

A monochromatic coloring is about one class only while rainbow coloring is about coloring with different colorsor partitioning into disjoint classes. Existence of a rainbow AP(3) has been shown. So, rainbow AP(3)'s exist undercertain conditions on color classes.

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