ON THE CENTRALITY OF GENERALIZED PETERSEN GRAPHS P(N,M)

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ABSTRACT: *The eccentricity* $e(v)$ *of a vertex* v *in a connected graph* G *is the distance between* v *and a vertex farthest from* v *in* G. *The radius* $rad(G)$ *is the minimum eccentricity of the vertices of* G, *whereas the diameter* $diam(G)$ *is the maximum eccentricity. A vertex* v *is a central vertex if* $e(v) = rad(G)$. *The center* $Cen(G)$ *is the subgraph induced by the central vertices of* G. *We call* G *self centered if* $Cen(G) = G$. *In this paper, we establish that almost all members of the family of generalized Petersen graphs* $P(n,m)$ *are self centered.*

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1 INTRODUCTION

Determining the diameter of a circulant graph (or any graph for that matter) is a difficult problem. However, Boesh and Wang [2] were able to determine the minimum diameter among all circulants on n vertices and having two jump sizes.

The generalized Petersen graph $P(n,m)$ for $n\geq 3$ and $1\leq m\leq\left\lfloor\frac{n-1}{2}\right\rfloor$ is a graph consisting of an inner star polygon (circulant graph) and an outer regular polygon (C_n) with corresponding vertices in the inner and outer polygons connected with edges. $P(n,m)$ has $2n$ vertices with vertex set and $3n$ edges with an edge set $E=\{u_iu_{i+1}, u_iv_i, v_iv_{i+m} \mid 1\leq i\leq n\}$ where indices are taken modulo n.

It is important to know the exact location of an emergency facility such as a hospital or fire station, we minimize the response time (distance) between the facility and the location of a possible emergency. In determining the location for a service facility such as a post office, power station, or employment office, we want to minimize the total travel time for all people in the district. The construction of a railroad line, pipeline, or superhighway, demands the minimization of the distance from the new structure to each of the communities to be served. Each of these situations deals with the concept of centrality [2, 4].

The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u - v path in G. It is known that the distance function satisfies the following properties in any connected graph G.

1. $d(u,v)\geq 0$ for all $u,v\in V(G)$
2. $d(u,v)=0$ if and only if $u=v$
3. $d(u,v)=d(v,u)$ for all $u,v\in V(G)$
4. $d(u,w)\leq d(u,v)+d(v,w)$ for all $u,v,w\in V(G)$

The fact that the distance d satisfies properties 1-4 means that d is a metric and $(V(G),d)$ is a metric space [3, 4].

For a vertex v in a connected graph G, the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity of the vertices of G is its radius and the maximum eccentricity is its diameter. They are denoted by rad(G) and diam(G), respectively. A vertex v in G is a central vertex if $e(v)$ = rad(G). The subgraph Cen(G) induced by central vertices of G is called the center of G. If every vertex is a central vertex, then Cen(G)=G and G is called self centered. A vertex v in a connected graph G is called a peripheral vertex if $e(v)$ = diam(G). The subgraph of G induced by its peripheral vertices is the periphery of G and is denoted by Per(G).

Now we present a lemma that will be used in our subsequent work.

Lemma 1.1. *[3]*

The center of the graph is the full graph if and only if the radius and diameter are equal.

In this paper, we determine center and periphery for the family of generalized Petersen graphs $P(n,m)$ for all n and $m\geq 3$.

2 Center and Periphery for *P*(*n*,*m*)

The automorphism group of a graph is a group of adjacency preserving permutations of its vertices. The automorphism ρ permutes the vertices of $P(n,m)$ as $\rho(u_i)=u_{i+1}$ and $\rho(v_i)=v_{i+1}$.

Also if x and y are vertices of a connected graph G and $\phi\in Aut(G)$, then $d(x,y)=d(\phi(x),\phi(y))$ [1,5].Thus to determine the eccentricity of the vertices of $P(n,m)$, it is enough to determine the eccentricity of exactly one vertex on the inner and outer cycle.

Now we present some results regarding the centrality of members of the family of generalized Petersen graphs $P(n,1)$ and $P(n,2)$.

Theorem 2.1. [8]

All the members of the family of generalized Petersen graphs P(n,1) are self centered with

$$diam(P(n,1)) = rad(P(n+1))$$
$$=\begin{cases}\frac{n}{2}+1; \text{ if n is odd;}\\ \frac{n-1}{2}+1; \text{ if n is even.}\end{cases}$$

Theorem 2.2. [8]

All the members of the family of generalized Petersen graphs P(n,2), n≡0,2(mod 4) and n>6 are self centered with

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$$diam(P(n,2)) = rad(P(n+2))$$

$$= \begin{cases} \frac{n}{4}+2; \text{ if } n=4k; \\ \frac{n-2}{4}+3; \text{ if } n=4k+2. \end{cases}$$

Theorem 2.3*.[8]*

For the family of generalized Petersen graphs P(n,2), n□1,3(mod 4), n>7, the center is a subgraph of P(n,2) induced by the vertices on the inner cycle and periphery is a subgraph of P(n,2) induced by the vertices on the outer cycle. In particular none of the graphs P(n,2) with n□1,3(mod 4), n>7 are self centered.

Now we establish that $P(n,m)$ is self centered for the remaining values of m.

Theorem 2.4*.*

For generalized Petersen graph $P(n,m)$, and m□3

Cen($P(n,m)$)=$P(n,m)$

and

Per($P(n,m)$)=$P(n,m)$.

Proof:

Since $m<\frac{n}{2}$ for $P(n,m)$, it means that $2m<n$. By division algorithm there exist integers k and r such that $n=2mk+r$, 0□$r<2m$. Thus to find radius and diameter of $P(n,m)$ we characterize n into the residue classes modulo $2m$.

Let $n=2mk$ and m be an even number. Then there are (m□1), odd number of vertices between $v_{m(k-1)+1}$ and v_{mk+1}, the farthest vertex from v_1, a vertex on the inner cycle must exist between these two vertices see figure 1.

The distance from v_1 to $\frac{m}{2}$ vertices after $v_{m(k-1)+1}$ on the inner cycle increase from k+2 to $(k-1)+\frac{m}{2}+2$,

since

$d(v_1,v_{m(k-1)+1})=k-1$, $d(u_{m(k-1)+1},u_{m(k-1)+1+\frac{m}{2}})=\frac{m}{2}$,

the vertices

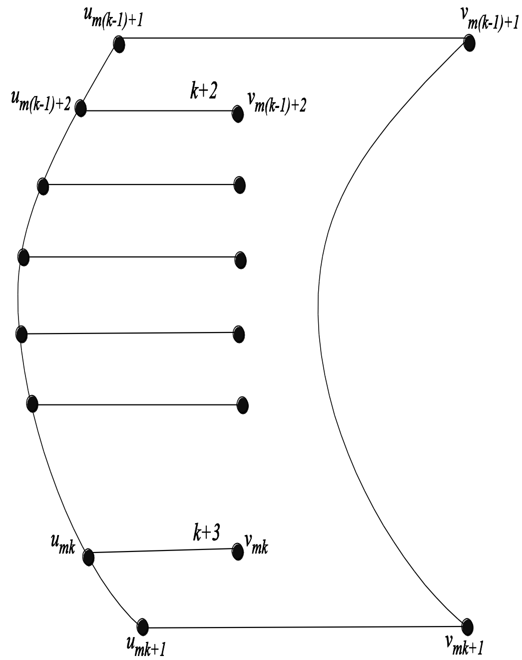
$v_{m(k-1)+1}$, $u_{m(k-1)+1}$ and $u_{m(k-1)+1+\frac{m}{2}}$, $v_{m(k-1)+1+\frac{m}{2}}$

are adjacent.

Therefore $d(v_1,v_{m(k-1)+1+\frac{m}{2}})=(k-1)+\frac{m}{2}+2$ ¾® (1)

The distance from v_1 to the remaining $\frac{m-2}{2}$ vertices before v_{mk+1} increase from k+3 to $k+\frac{m-2}{2}+2$.

Thus $d(v_1,v_{mk+1-(\frac{m-2}{2})})=k+\frac{m-2}{2}+2$ ¾® (2).



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The distance from v_1 to the vertices on the outer cycle

$d(v_1,u_{m(k-1)+1+\frac{m}{2}})=(k-1)+\frac{m}{2}+1$ ¾® (3).

$d(v_1,u_{mk+1-(\frac{m-2}{2})})=k+\frac{m-2}{2}+1$ ¾® (4)

From (1), (2), (3) and (4) it follows that $v_{m(k-1)+1+\frac{m}{2}}$ and $v_{mk+1-(\frac{m-2}{2})}$ both are farthest from v_1.

Thus $e(v_1)=k+\frac{m-2}{2}+2$ ¾® (A).

Now we determine the eccentricity of u_1, a vertex on the outer cycle. Since u_1 is adjacent to v_1, from (3) and (4) we have

$d(u_1,u_{m(k-1)+1+\frac{m}{2}})=(k-1)+\frac{m}{2}+2$ ¾® (5).

$d(u_1,u_{mk+1-(\frac{m-2}{2})})=k+\frac{m-2}{2}+2$ ¾® (6).

To determine the distance from u_1 to the vertices on the inner cycle, we first prove that $d(u_i,v_j)=d(u_j,v_i)$ for all i,j. Since d is a metric on the vertices of G. Therefore

$d(u_i,v_j)=d(u_i,v_i)+d(v_i,v_j)=d(u_j,v_j)+d(v_i,v_j)=d(u_j,v_i)$

From (3) and (4) by using this result we have

$d(u_1,v_{m(k\text{-}1)+1+\frac{m}{2}})=(k\text{-}1)+\frac{m}{2}+1$ ¾® (7).

$d(u_1,v_{mk+1\text{-}(\frac{m\text{-}2}{2})})=k+\frac{m\text{-}2}{2}+1$ ¾® (8)

It follows from (5), (6), (7) and (8) that $u_{m(k\text{-}1)+1+\frac{m}{2}}$ and $u_{mk+1\text{-}(\frac{m\text{-}2}{2})}$ both are farthest from u_1.

Thus $e(u_1)=k+\frac{m\text{-}2}{2}+2$ ¾® (B).

From (A) and (B) each vertex on the inner and outer cycle have same eccentricity.

Consequently, each vertex is a central as well as peripheral vertex.

Let $n=2mk$ and m be an odd number. Then there are (m□1), even number of vertices between $v_{m(k\text{-}1)+1}$ andv_{mk+1}, the farthest vertex from v_1 must exist between these two vertices.

The distance from v_1 to $\frac{m\text{-}1}{2}$ vertices after $v_{m(k\text{-}1)+1}$ on the inner cycle increase from k+2 to $(k\text{-}1)+\frac{m\text{-}1}{2}+2$,

Since

$$d(v_1,v_{m(k\text{-}1)+1})=k\text{-}1 \quad ,$$

$$d(u_{m(k\text{-}1)+1},u_{m(k\text{-}1)+1+\frac{m\text{-}1}{2}})=\frac{m\text{-}1}{2},$$

the vertices $v_{m(k\text{-}1)+1}$, $u_{m(k\text{-}1)+1}$ and $u_{m(k\text{-}1)+1+\frac{m\text{-}1}{2}}$, $v_{m(k\text{-}1)+1+\frac{m\text{-}1}{2}}$

are adjacent.

Therefore $d(v_1,v_{m(k\text{-}1)+1+\frac{m\text{-}1}{2}})=(k\text{-}1)+\frac{m\text{-}1}{2}+2$ ¾® (1).

The distance from v_1 to the remaining $\frac{m\text{-}1}{2}$ vertices before v_{mk+1} increase from k+3 to $k+\frac{m\text{-}1}{2}+2$.

Thus $d(v_1,v_{mk+1\text{-}(\frac{m\text{-}1}{2})})=k+\frac{m\text{-}1}{2}+2$ ¾® (2).

The distance from v_1 to the vertices on the outer cycle

$d(v_1,u_{m(k\text{-}1)+1+\frac{m\text{-}1}{2}})=(k\text{-}1)+\frac{m\text{-}1}{2}+1$ ¾® (3).

$d(v_1,u_{mk+1\text{-}(\frac{m\text{-}1}{2})})=k+\frac{m\text{-}1}{2}+1$ ¾® (4)

From (1), (2), (3) and (4) it follows that $v_{mk+1\text{-}(\frac{m\text{-}1}{2})}$ is farthest from v_1.

Thus $e(v_1)=k+\frac{m\text{-}1}{2}+2$ ¾® (A).

Now we determine the eccentricity of u_1, a vertex on the outer cycle. Since u_1 is adjacent to v_1, from (3) and (4) we have

$d(u_1,u_{m(k\text{-}1)+1+\frac{m\text{-}1}{2}})=(k\text{-}1)+\frac{m\text{-}1}{2}+2$ ¾® (5).

$d(u_1,u_{mk+1\text{-}(\frac{m\text{-}1}{2})})=k+\frac{m\text{-}1}{2}+2$ ¾® (6).

To determine the distance from u_1 to the vertices on the inner cycle, we use $d(u_i,v_j)=d(u_j,v_i)$ for all i,j.

From (3) and (4) by using this result we have

$d(u_1,v_{m(k\text{-}1)+1+\frac{m\text{-}1}{2}})=(k\text{-}1)+\frac{m\text{-}1}{2}+1$ ¾® (7).

$d(u_1,v_{mk+1\text{-}(\frac{m\text{-}1}{2})})=k+\frac{m\text{-}1}{2}+1$ ¾® (8)

It follows from (5), (6), (7) and (8) that $u_{mk+1\text{-}(\frac{m\text{-}1}{2})}$ is farthest from u_1.

Thus $e(u_1)=k+\frac{m\text{-}1}{2}+2$ ¾® (B).

From (A) and (B) each vertex on the inner and outer cycle have same eccentricity.

Consequently, each vertex is a central as well as peripheral vertex.

For the remaining n or $n=2mk+r$, 0□$r<2m$, the vertex farthest from v_1 can be determined by finding out the distances from v_1 to $v_{\frac{n}{2}+1}$ and $v_{\frac{n+1}{2}}$ for even and odd n respectively. It is easy to see that n is even or odd according as r is even or odd.

Thus for all r, 2□r□2m□2

Implies that

$mk+1$£$mk+\frac{r}{2}+1$£$mk+m$, $mk+2$£$\frac{n}{2}+1$£$mk+m$ for even n

and

$mk+1$£$\frac{n+1}{2}<mk+m$ for odd n.

We also observe that the farthest vertex from v_1 can be evaluated by using at least one of the following shortest path from v_1.

$d(v_1,v_{m(k\text{-}1)+1})=k\text{-}1$, $d(v_1,v_{mk+1})=k$,

and

$$d(v_1,v_{m(k\text{-}1)+r+1})=k+1$$

whenever 1□$r<2m$□1.

Hence

$$\mathrm{Cen}(P(n,m))=P(n,m)$$

and

Per($P(n,m)$)=$P(n,m)$ for all n and m□3.

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