APPLICATIONS OF OPTIMAL HOMOTOPY ASYMPTOTIC METHOD TO HEAT TRANSFER PROBLEMS

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ABSTRACT: *In this paper we have applied Optimal Homotopy Asymptotic Method (OHAM) to two models of Boundary Value Problems (BVPs). The results obtained by OHAM are compared with the exact solution. The obtained solutions show that OHAM is effective, simpler easier and explicit.*

Keywords: OHAM, heat transfer, Non-linear differential equations

1. INTRODUCTION

In engineering and applied sciences most of the problems are nonlinear in nature. Some of the analytic methods are given in the literature. The researchers then searched new methods like Adomian Decomposition Method (ADM) [1], Variational Iterative Method (VIM) [2] , Differential Transform Method (DTM) [3], Group Analysis Method [4], Homotopy Perturbation Method (HPM) [5], Radial Basis Function [6], and These methods are treated for weakly nonlinear problems and limited for strongly nonlinear problems. The perturbation methods were introduced for strongly nonlinear BVPs [7-9]. A small parameter is involved in these methods and is difficult to find. For this purpose Artificial Parameters Method [10], Homotopy Analysis Method (HAM) [11] and Homotopy Perturbation Method (HPM) [5] have been introduced. These methods combined the homotopy with the perturbation techniques. Recently, Vasile Marinca *et al* introduced OHAM for nonlinear BVPs. This method does not require small parameter to be assumed [12-16].

The applicability of these methods is studied in [17-24]. The motivation of this paper is to apply OHAM for the solution of nonlinear BVPs arising in heat transfer.

In section 2, the basic idea of OHAM is formulated [12-16]. In Section 3, the effectiveness of the OHAM formulation for BVPs arising in heat transfer has been studied. Three special cases of nonlinear boundary value problems have been analyzed [25-26].

2. Fundamental Mathematical Structure of OHAM

We consider a general nonlinear problem:

$$\mathcal{L}\left(\varsigma\left(z\right)\right)+k\left(z\right)+\mathcal{N}\left(\varsigma\left(z\right)\right)=0, \tag{2.1}$$

with

$$\mathcal{B}\left(\varsigma,\frac{d\varsigma}{dz}\right)=0. \tag{2.2}$$

where $\mathcal{L}$ is the linear operator, $\varsigma\left(z\right)$ unknown function, $k\left(z\right)$ known function, $\mathcal{N}\left(\varsigma\left(z\right)\right)$ nonlinear operator and $\mathcal{B}$ a boundary operator.

According to OHAM, we can take $\tau\left(z,s\right)$: $\Omega\times\left[0,1\right]\rightarrow\Re$ satisfying

$$\left(1-s\right)\left[\mathcal{L}\left(\tau\left(z,s\right)\right)+k\left(z\right)\right]=$$
$$F\left(s\right)\left[\mathcal{L}\left(\tau\left(z,s\right)\right)+f\left(z\right)+\mathcal{N}\left(\tau\left(z,s\right)\right)\right], \tag{2.3}$$

$$\mathcal{B}\left(\tau\left(z,s\right),\frac{\partial\tau\left(z,s\right)}{\partial z}\right)=0, \tag{2.4}$$

where $s\in\left[0,1\right]$ is embedding parameter, $\tau\left(z,s\right)$ unknown function, $F\left(s\right)\neq 0$ auxiliary function. Also

$$s=0\Rightarrow\tau(z,0)=\varsigma_0\left(t\right), \tag{2.5}$$

$$s=1\Rightarrow\tau\left(z,1\right)=\varsigma\left(z\right), \tag{2.6}$$

From Eq. (2.1) and (2.2) we have

$$\mathcal{L}\left(\varsigma_0(z)\right)+k(z)=0,\quad \mathcal{B}\left(\varsigma_0,\frac{d\varsigma_0}{dz}\right)=0. \tag{2.7}$$

Choosing

$$F\left(s\right)=sB_1+s^2B_2+s^3B_3+... \tag{2.8}$$

Where $B_1,B_2,B_3,...$ are optimal constants to determine.

Expanding $\tau\left(z,s,B_i\right)$ by Taylor's series in the form,

$$\tau\left(z,s,B_i\right)=\varsigma_0\left(z\right)+\sum_{k=1}^{\infty}\varsigma_k\left(z,B_i\right)s^k,\quad i=1,2,... \tag{2.9}$$

Substituting Eq. (2.9) into Eq. (2.1) and Eq. (2.2) and equating the coefficient of same powers of s , we get

$$\mathcal{L}\left(\varsigma_1(z)\right)=B_1\mathcal{N}_0\left(\varsigma_0(z)\right), \qquad \mathcal{B}\left(\varsigma_1,\frac{d\varsigma_1}{dz}\right)=0 , \tag{2.10}$$

$$\mathcal{L}\left(\varsigma_2(z)\right)-\mathcal{L}\left(\varsigma_1(z)\right)=B_1\mathcal{N}_0\left(\varsigma_0(z)\right)+$$
$$B_1\left[\mathcal{L}\left(\varsigma_1(z)\right)+\mathcal{N}_1\left(\varsigma_0(z),\varsigma_1(z)\right)\right], \qquad \mathcal{B}\left(\varsigma_2,\frac{d\varsigma_2}{dz}\right)=0, \tag{2.11}$$

$$\mathcal{L}\left(\varsigma_k(z)\right)-\mathcal{L}\left(\varsigma_{k-1}(z)\right)=B_k\mathcal{N}_0\left(\varsigma_0(z)\right)+$$
$$\sum_{i=1}^{k-1}B_i\left[\mathcal{L}\left(\varsigma_{k-i}(z)\right)+\mathcal{N}_{k-i}\left(\varsigma_0(z),\varsigma_1(z),...,\varsigma_{k-i}(z)\right)\right],$$
$$\mathcal{B}\left(\varsigma_k,\frac{d\varsigma_k}{dz}\right)=0, k=2,3,..., \tag{2.12}$$

In Eq. (2.12) $\mathcal{N}_{k-i}\left(\varsigma_0(z),\varsigma_1(z),...,\varsigma_{k-i}(z)\right)$ is the coefficient of s^{k-i} in the expansion.

$$\mathcal{N}\left(\tau\left(z,s,B_i\right)\right)=\mathcal{N}_0\left(\varsigma_0(z)\right)+$$
$$\sum_{k\geq1}\mathcal{N}_k\left(\varsigma_0,\ \varsigma_1,\ ...\varsigma_k\right)s^k, i=1,2,3,... \tag{2.13}$$

It is to be noted that u_k , $k\geq0$ gives a linear BVP which can be easily solved.

The convergence of the series (2.9) depends upon the auxiliary constants $B_1,B_2,...$. For $s=1$

$$\tilde{\varsigma}\left(z,B_i\right)=\varsigma_0(z)+\sum_{k\geq1}\varsigma_k\left(z,B_i\right). \tag{2.14}$$

Putting Eq. (2.14) into Eq. (2.1), we obatin

$$R\left(z,B_i\right)=\mathcal{L}(\tilde{\varsigma}(z,B_i))+k(z)+\mathcal{N}(\tilde{\varsigma}(z,B_i)). \tag{2.15}$$

If $R\left(z,B_i\right)=0$ then $\tilde{\varsigma}\left(z,B_i\right)$ gives the exact solution of the problem which does not happen in case of nonlinear problems.

For the computation of C_i , different methods like Galerkin's Method, Ritz Method, Least Squares Method and Collocation Method are used. By the Method of Least Squares

$$I\left(B_i\right)=\int_a^b R^2\left(z,B_i\right)dz, \tag{2.16}$$

where a and b are two constants.

The auxiliary constants B_i , $i=1,2,...,m$ can be found by

$$\frac{\partial I}{\partial B_1}=\frac{\partial I}{\partial B_2}=...=\frac{\partial I}{\partial B_m}=0. \tag{2.17}$$

The m th order approximate solution is obtained by these constants. The constants B_i can also be found

$$R\left(k_1,B_i\right)=R\left(k_2,B_i\right)=...=R\left(k_m,B_i\right)=0,\ i=1,2,...,m. \tag{2.18}$$

The auxiliary function $H\left(p\right)$ is useful for convergence and error minimization.

3. Application of OHAM to heat transfer problems

The application of OHAM is studied in the following models.

Model 1 [25]

Consider a lumped system BVP of the form

$$\left(1+\beta_1\xi\right)\frac{d\xi}{dz}+\xi+\beta_2\xi^4=0,$$
$$\xi\left(0\right)=1. \tag{3.1.1}$$

Zeroth Order Problem: $\xi_0+\xi_0'=0,\quad \xi_0\left(0\right)=1,$ (3..1.2)

its solution is

$$\xi_0=\mathrm{e}^{-z}. \tag{3.1.3}$$

First Order Problem:

$$\xi_1'=\left(\xi_0-\xi_1\right)+C_1\left(\xi_0{}^4+\xi_0\xi_0'\right),\ \xi_1\left(0\right)=0, \tag{3.1.4}$$

whose solution is

$$\xi_1\left(z,B_1\right)=-\frac{2B_1}{3}e^{-z}+B_1e^{-2z}-\frac{B_1}{3}e^{-4z}. \tag{3.1.5}$$

Second Order Problem:

$$\xi_2'=\left(\xi_1+\xi_1'-\xi_2\right)+B_1\left(4\xi_1\xi_0{}^3+\xi_1\xi_0'+\xi_0\xi_1'\right)+B_2\xi_0{}^4,$$
$$\xi_2\left(0\right)=0, \tag{3.1.6}$$

we get the following solution

$$\xi_2\left(z,B_1,B_2\right)=\frac{1}{36}\left(10B_1{}^2-48B_2\right)$$
$$-\frac{1}{36}\left(71B_1{}^2+48B_2+24B_1\right)e^{-z}$$
$$-\frac{1}{36}\left(38B_1{}^2-60B_1-12B_2\right)e^{-2z}$$
$$-\frac{1}{36}\left(22B_1{}^2-24B_1-24B_2\right)e^{-3z}$$
$$-\frac{1}{36}\left(76B_1{}^2-12B_1-12B_2\right)e^{-4z}$$
$$+\frac{13}{36}e^{-5z}+\frac{32}{36}B_1{}^2e^{-6z}+\frac{8}{36}B_1{}^2e^{-7z}. \tag{3.1.7}$$

From Eqs. (3.1.3), (3.1.5), and (3.1.7), we obtain

$$\xi\left(z,B_1,B_2\right)=\xi_0+\xi_1\left(z,B_1\right)+\xi_2\left(z,B_1,B_2\right). \quad (3.1.8)$$

$$\xi\left(z\right)=C_0+C_1e^{-z}+C_2e^{-2z}+C_3e^{-3z}+C_4e^{-4z}+C_5e^{-5z}+C_6e^{-6z}+C_7e^{-7z}, \quad (3.1.9)$$

where

$$C_0=\frac{1}{36}\left(10B_1^2-48B_2\right),$$
$$C_1=-\frac{1}{36}\left(71B_1^2+48B_2+48B_1\right),$$
$$C_2=-\frac{1}{36}\left(38B_1^2+12B_2+24B_1\right),$$
$$C_3=-\frac{1}{36}\left(22B_1^2+24B_2+24B_1\right),$$
$$C_4=-\frac{1}{36}\left(76B_1^2+12B_2+24B_1\right),$$
$$C_5=\frac{13}{36}B_1^2,$$
$$C_6=\frac{32}{36}B_1^2, C_7=\frac{8}{36}B_1^2. \quad (3.1.10)$$

Using Eq. (3.1.9) in (3.1.1) and applying the method of least square, we obtain

$C_1=-0.6432739673843123$ and $C_2=0.3131576613661234$.

Substituting these values in Eq. (3.1.9) for $\beta_1=1=\beta_2$, we obtain

$$\xi\left(z\right)=1.2045e^{-z}+0.264785e^{-2z}-1.261115e^{-3z}-1.02716e^{-4z}-0.591555e^{-5z}-0.01e^{-6z}+0.0870e^{-7z}. \quad (3.1.11)$$

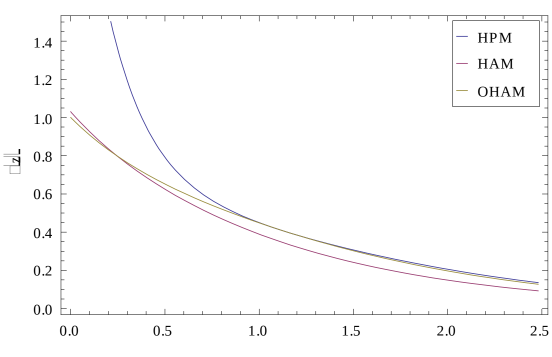

Fig. 1. Comparison of HPM, HAM and OHAM of Eq. (3.2.3) for $\beta_1=1=\beta_2$.

Model 2 [25]

By assuming $\beta_1=0,\beta_2=\lambda$ in Eq. (3.1.1), we obtain

$$\frac{d\xi}{dz}+\xi+\lambda\xi^4=0, \quad \xi\left(0\right)=1. \quad (3.2.1)$$

Zeroth Order Problem: $\xi_0+\xi_0'=0, \quad \xi_0\left(0\right)=1,$ (3.2.2)

its solution is

$$\xi_0=e^{-z}. \quad (3.2.3)$$

First Order Problem:

$$\xi_1'=B_1\xi_1'+\left(\xi_0-\xi_1\right)+B_1\left(\xi_0+\lambda\xi_0^4\right)+\xi_0',$$
$$\xi_1\left(0\right)=0, \quad (3.2.4)$$

whose solution is

$$\xi_1\left(z,B_1\right)=\frac{\lambda B_1}{3}\left(e^{-z}-e^{-4z}\right). \quad (3.2.5)$$

Second Order Problem:

$$\xi_2'=\left(1+B_1\right)\xi_1'+\left(\xi_1-\xi_2\right)+B_1\xi_1\left(1+4\lambda\xi_0^3\right)+B_2\left(\xi_0+\xi_0'+\lambda\xi_0^4\right), \quad \xi_2\left(0\right)=0, \quad (3.2.6)$$

its solution is

$$\xi_2\left(z,B_1,B_2\right)=\left(\frac{B_1^2}{3}-\frac{B_1}{3}+\frac{2B_1}{9}-\frac{B_2}{3}-\frac{2\lambda B_1^2}{9}\right)\lambda e^{-4z}+\left(\frac{B_1}{3}-\frac{B_1^2}{3}+2B_1^2\lambda+3B_2\right)\lambda e^{-z}-\frac{2}{9}\lambda^2B_1^2e^{-7z}. \quad (3.2.7)$$

From Eqs. (3.2.3), (3.2.5), and (3.2.7), we obtain:

$$\xi\left(z,B_1,B_2\right)=\xi_0+\xi_1\left(z,B_1\right)+\xi_2\left(z,B_1,B_2\right). \quad (3.2.8)$$

Using Eq. (3.2.8) in (3.2.1) and applying the technique as discussed in Eqs. (2.15)- (2.17), we obtain

$C_1=-0.610505203261472$ and $C_2=-0.017505005737890128$.

Substituting these values in Eq. (3.2.8), we have

$$\xi\left(z\right)=\left(1-\left(0.149257492\right)\lambda\right)e^{-z}+0.025881176\lambda e^{-3z}+0.278803961\lambda e^{-4z}+0.155463297\lambda e^{-7z}. \quad (3.2.9)$$

4. RESULTS AND DISCUSSIONS

In section 2, OHAM formulation is given and applied to two models of BVPs. For most of the computation, we have used Mathematica 7. Fig. 1 shows that OHAM and HAM solutions are similar and Fig. 2 shows that OHAM is same to NHPM solutions for various values of λ and Fig.3 shows that OHAM and HPM solutions are nearly alike [25].

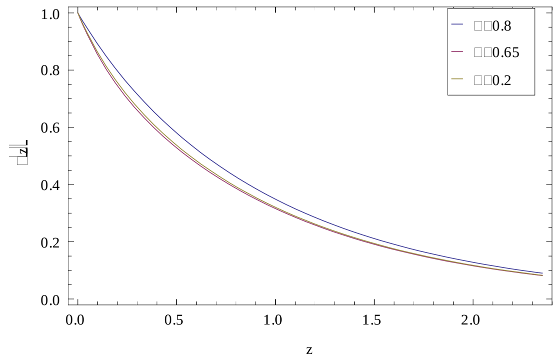

Fig. 2. Plot of $\xi(z)$ for different values of λ

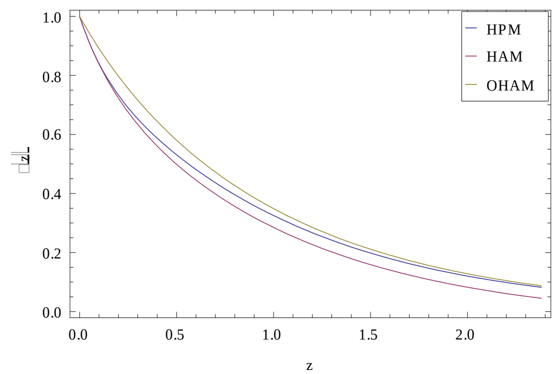

Fig. 3. Comparison of HPM, HAM and OHAM of Eq. (3.2.1) for $\lambda = 1$

5. CONCLUSION

In this paper, we have proved that OHAM is simpler, can easily to control the convergence and has less computational work. Therefore, OHAM is valid and great potential for the solution of BVPs.

6. REFERENCES

1. Chowdhury S.H., A comparison between the modified homotopy perturbation method and adomian decomposition method for solving nonlinear heat transfer equations, *J. Appl. Sci*., **11** (8), 1416-1420 (2011).
2. Chun C., Fourier series based variational iteration method for a reliable treatment of heat equation with variable coefficient, Int. J. nonlinear. Sci. Num. Simu., **10**, 1383-1388 (2007).
3. Ganji D.D, Afrouzi R. A., The application of differential transformation method to nonlinear equation arising in heat transfer, Int. Comm. *Heat Mass Trans*., **38** (6), 815-820 (2011)
4. Ayub M, Haq S, Siddiqui A.M, Hayat T., Group analysis of two dimensional flow of a non-Newtonian fluid, *I. J. Sci*., **13**(1), 47-59 (2003)
5. He J.H., Homotopy perturbation technique, Comp. Math. *Mech. Engg*., **178**, 257-262 (1999)
6. Ganji D.D., The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Phy. Lett., A. **355**, 337-341 (2006)
7. Bellman R., Perturbation techniques in Mathematics, Phys. Engg. Holt. New York 1964
8. Cole J.D., Perturbation methods in applied mathematics, Blais. Walth. MA. 1968
9. O'Malley R.E., Introduction to singular perturbation, Acad. Press. New York. 1974
10. Liu G.L., New research direction in singular perturbation theory artificial parameter approach and inverse perturbation technique Math. Mech. 1997
11. Lio S.J., The proposed homotopy analysis technique for the solution of nonlinear problems, *PhD Thesis*. Shan. Jio. Tong. Uni. 1992
12. Marinca V, Herisanu N., Optimal homotopy asymptotic method with application to thin film flow. *Cent. Euro. J. Phys*., **6**, 648–53 (2008)
13. Marinca V, Herisanu N., Application of optimal homotopy asymptotic method for solving nonlinear equation arising in heat transfer, Int. Comm. *Heat. Mass. Trans*., **35**, 710–715 (2008)
14. Marinca V, Herisanu N., An optimal homotopy asymptotic method applied to the steady flow of a fourth grade fluid past a porous plate, Appl. Math. Lett. **22**, 245-251 (2009)
15. Marinca V, Herisanu N, Names I., A new analytic approach to nonlinear vibration of an electrical machine. *Proc. Roma. Acad*., **9**, 229-236 (2008)
16. Marinca V, Herisanu N., Determination of periodic solutions for the motion of a particle on a rotating parabola by means of optimal homotopy asymptotic method, *J. Sound. Vibr*., **11,** 001-005 (2009)
17. Idrees M, Islam, Tirmizi A.M., Application of optimal homotopy asymptotic method of the Korteqag de varies equation, Comp. Math. Appl. 63, 695-707 (2012)
18. Nawaz R, Ullah H, Islam S, Idrees M., Application of optimal homotopy asymptotic method to Burgers equation, *J. Appl. Math*., Vol. Art ID 387478, 8 pages (2013)
19. M Idees, Islam S, Haq S., Application of optimal homotopy asymptotic method to eight order initial and boundary value problem, *Appl. Math. Comp*., **4**, 73-80 (2010)
20. Iqbal S, Idrees M, Siddiqui A.M., Some application of linear and nonlinear Klein Gorden equation using optimal homotopy asymptotic method, *J. Appl. Math. Comp*., **216**, 2898-2909 (2010)
21. H. Ullah, S. Islam, M. Idrees, R. Nawaz, Optimal Homotopy Asymptotic Method to doubly wave solutions of the coupled Drinfel'd-Sokolv-Wilson equations, Math. Prob. Engg. Arti. ID 362816, 8 pages.
22. H. Ullah, S. Islam, M. Idrees and M. Arif, Solution of Boundary Layer Problems with Heat Transfer by Optimal Homotopy Asymptotic Method. Abst. App. Anal.Volu. 2013 (2013), Arti. ID 324869, 10 pages.

23. Idrees M, Islam S, Haq S., Application of optimal homotopy asymptotic method to squeezing flow, Comp. Math. Appl., **59**, 3858-3866 (2010)

24. Khan A, Islam S., Application of HPM to vector host epidemic model with nonlinear incidences, Reas. J. Rec. Sci., **2** (6), 90-95 (2013)

25. Caruntu B, Bota C., Approximate polynomial solutions for the nonlinear het transfer problems using the squared remainder minimization method. Int. J. Heat. Mass. Tran., **39**, 1336-1341 (2012)