IRREDUCIBILITY OF POLYNOMIALS IN THE FIELD OF RATIONAL NUMBERS

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Irreducibility of the polynomials and their reciprocal polynomials has been a very interesting and difficult discussion in the literature of mathematics in past and is still in present . is the polynomial

$f(x) = (x - a_1)(x - a_2) \ldots\ldots\ldots (x - a_n) - 1$ ………………..(1)

irreducible in the realm of rational integers , assuming a_1 , a_2 , a_3 ,…….a_n are distinct integers ? This is the objective of this paper .

First of all this question was raised by I . Schur (1) in 1906 . to the question of irreducibility of the polynomials of the form

$f(x) = ax(x - a_2)(x - a_2) \ldots\ldots\ldots (x - a_n) \pm c$ ……………(2)

and variants there of . In particular Schultz's conditions for the reducibility of the polynomials of the form

$f(x) = ax(x - a_2) \ldots\ldots\ldots (x - a_p) \pm p$

(where a is a square and a_j are neither congruent to zero nor to each other, are given .

Westlund (2) proved that f (x) is irreducible in the above domain of rational numbers and that

$g(x) = (x - a_1)(x - a_2) \ldots\ldots\ldots (x - a_n) + 1$

can be reduce only if it is a perfect square in which case n must be even .

The following three theorems are due to Seres (3)

.1) If p (x) is a monic polynomial whose zeroes are distinct , rational integers ,
then the polynomial

$f(x) = (p(x))^2 + 1$

is irreducible in the field of rational numbers

2) If ϕ (x) is the cyclotomic polynomial of order m , and p (x) is a monic polynomial of degree greater than four whose zeros are distinct , rational integers .
Then $\phi_m(p(x))$ is irreducible in the field of rational numbers . If the degree of p (x) is less than 5 , the same result holds with some exceptions

3) Let p (x) be a monic polynomial with rational , integer coefficients and of degree less than that of $\phi_m(p(x))$. If $R(x) = P(x)Q(x)$, then $\phi_m(R(x))$ is irreducible in the field of rational numbers .

Seres (3) also proved the following theorem :

Theorem . Let $\phi_m(x)$ be the cyclotomic polynomial of order m >. 2 and **n**
let $P(x) = \prod_{k=1}^{n} (x - a_k)$, where the a_k are distinct , rational integers .
Then $\phi_m(p(x))$ is irreducible over the field of rational numbers .

Also $P(x) = \prod_{K=1} (x - a_k)$
Where (k , m) = 1 is irreducible in the cyclotomic field of order m , except when m = 12 and
$P(x) = (x - n)^3 - (x - n)$.

I , then , in a different way , proved the irreducibility criteria for variants of the above type of polynomials in my research thesis of Ph . D .

THEOREM 1

. Let $f(x) \in Z[x]$ where Z is the ring of rational integers . Assume deg f = n .
Let $|f(0)| > 1$ while $\{c_1, c_2, \ldots\ldots\ldots c_r\}$ is the set of all divisors of f (0) of absolute value greater than 1 . Suppose there exist a_1 , a_2 , …… , $a_n \in Z$ ($a_i \neq a_j$ for $i \neq j$)
Such that

(a) a_k does not divide $c_j \pm 1$ (k = 1 , 2 , 3 , ………., n ; j = 1 , 2 , ……., r)

(b) $|a_k| >. 2$ (k = 1 , 2 , 3 , …….., n) ;

(c) $f(a_k) = P_k$ is a rational , prime integer (k = 1 , 2 , ……., n) .

Then f (x) is irreducible in Q , the field of rational numbers .

PROOF :

Suppose $f(x) = f_1(x) f_2(x)$
Where $f_1(x)$, $f_2(x) \in Z[x]$, $f_1(0) = b_1$, $f_2(0) = c_1$

Case 1 . $|b_1| = 1$, $|c_1| = |f(0)| >. 1$. **Suppose**
$f_1(a_k) = \pm p_k$, $f_2(a_k) = \pm 1$
for any k (k = 1 , 2 , , ……. , n) . Contradiction of (a) , since a_k does not divide $c_1 \pm 1$
k = (1 , 2 , 3 , ……. , n)
Suppose $f_1(a_k) = -b_1$ for some k . (k = 1 , 2 , …….., n)
A contradiction of (b)
Therefore , $f_1(a_k) = b_1$ (k = 1 , 2 , ………., n) 3
Implying that $f_1(x) \equiv b_1$

Case II : $|b_1| >. 1$, $|c_1| .> 1$.
Without loss of generality, suppose that
$f_1(a_k) = \pm p_k$, $f_2(a_k) = \pm 1$
For some k (k = 1 , 2 , ………, n) , a contradiction of (a)

THEOREM 2 :

Let f (x) $\in Z[x]$, deg f = n
Let $f(0) = \pm p$ (p >. 0) where p is a rational prime . Furthermore, suppose there exist
[n/2] + 1 distinct integers a_1 , a_2 , ………., $a_{[n/2]+1}$ such that

a) a_k does not divide $p \pm 1$ (k = 1 , 2 , ……. , [n / 2] + 1) ;
b) $f(a_k) = p$ (k = 1 , 2 , ………., [n / 2] + 1)
c) p is odd or n .> 3

Then f (x) is irreducible over Q , the field of rational numbers .

PROOF :

Suppose $f(x) = f_1(x)\, f_2(x)$, where $f_1(x)$, $f_2(x) \in Z[x]$

$\mathbf{f_1(0)} = b_1$, $f_2(0) = c_1$

Without loss of generality , we may assume that deg $f_1 \le [n/2]$.

Case 1 . $|b_1| = 1$, $|c_1| = p$.

Suppose $f_1(a_k) = \pm p$, $f_2(a_k) = \pm 1$

For some k $(k = 1, 2, \ldots, [n/2] + 1)$.

Suppose $f_1(a_k) = -b_1$, for some k $(k = 1, 2, \ldots [n/2] + 1)$.

Contradiction of (a) . , since in this case , a_k would divide 2 .

Therefore, $f_1(a_k) = b_1$ $(k = 1, 2, \ldots [n/2] + 1)$

This implies that $f_1(x) \equiv b_1$

Case II : $|b_1| = p$, $|c_1| = 1$

Suppose

$f_1(a_k) = \pm 1$, $f_2(a_k) = \pm p$

for some k $(k = 1, 2, \ldots, [n/2] + 1)$

It is a contradiction of (a)

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Suppose

$f_1(a_k) = -b_1$, $f_2(a_k) = -c_1$

for some k $(k = 1, 2, \ldots, [n/2] + 1)$

It is a contradiction of (a) .

Therefore , $f_1(a_k) = b_1$ $(k = 1, 2, \ldots, [n/2] + 1)$

Implying that $f_1(x) \equiv b_1$

COROLLARY :

Let $f(x) = A(x - c_1)(x - c_2) \ldots (x - c_{2m}) \pm p$; $m \ge 4$ and A , a positive , square integer ;

$C_1 = 0 < c_2 < \ldots < c_{2m}$, each c_j an integer; p a prime .

Furthermore , let $A' = \sqrt{A}$, $a_{2r-1} = c_{4r-3}$, $a_{2r} = c_{4r}$, , $b_{2r-1} = c_{4r-2}$, $b_{2r} = c_{4r-1}$ and $p = A' b_1 \ldots b_m \pm 1$. Then the roots of $f(x)$ are real and distinct , and there is precisely one root in each of the following intervals .

$[a_j - 1/2 , a_{j+1/2}]$ $(j = 1, 2, \ldots, m)$

$[b_j - 1/2 , b_j + 1/2]$ $(j = 1, 2, \ldots m)$

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