USE OF MODIFIED EQUATION TO EXAMINE THE STABILITY OF UPSTREAM DIFFERENCING SCHEME FOR INITIAL VALUE PROBLEMS

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ABSTRACT: In this paper we investigate the quantitative behavior of a wide range of numerical methods for solving linear partial differential equations [PDE's]. In order to study the properties of the numerical solutions, such as accuracy, consistency, and stability, we use the method of modified equation, which is an effective approach. To determine the necessary and sufficient conditions for computing the stability, we use a truncated version of modified equation which helps us in a better way to look into the nature of dispersive as well as dissipative errors. The Wave Equation arises in the construction of characteristic surfaces for hyperbolic partial differential equations, in the calculus of variations, in some geometrical problems and in simple modals for gas dynamics, whose solution involves the method of characteristics. Accuracy and stability of Upstream Scheme is checked by using Modified Differential Equations [MDE's].

Key words: Accuracy, Stability, Modified Equation, Dispersive error, Upstream Differencing Scheme.

1. INTRODUCTION

To examine the simple linear partial differential equation with the help of modified equation, we consider 1st order Wave Equation. This equation describes the temperature distribution in a bar as a function of time. For converting this simple PDE into a modified equation, we use finite difference approximations by using the initial value conditions. This is obtained by expanding each term of finite difference approximation into a Taylor series, excluding time derivative, time - space derivatives higher than first order. Terms occurring in this MDE represent a sort of truncation error. These permit the order of stability and accuracy. With this approach, a modified equation, which is an approximating differential equation that is a more accurate model of what is actually solved numerically by the use of given numerical schemes. This equation describes the evolution of the scalar field a(x , t) carried along by flow at constant speed C. The solution is

$$a(x, t) = f(x - ct)$$

where f is determined from the initial condition

$$a(x, t = 0) = f(x).$$

2. MATERIAL AND METHODS

2.1 Modified Equation

Modified Equation [1] uses to examine the accuracy and stability of the solution originally solved by PDE's. This is obtained by expanding each term of finite difference equation with the help of Taylor Series. The general technique of developing modified equation for PDE's is presented by Warming and Hyett [2]. We know that the general Linear PDE [3] is represented as

$$\partial f/\partial t + \mathcal{L}_x(f) = 0$$

where $\mathcal{L}_x(f)$ is a linear spatial differential operator, f is a function of a spatial variable x. A more specific example of this is adviction equation,

$$\frac{\partial f}{\partial t} + c\left(\frac{\partial f}{\partial x}\right) = 0$$

where "c" is a real constant. The modified equation is used to deal with the numerical solution's behavior.

2.2 Difference Approximation

We are using here forward difference approximations [4]

$$f_t = \frac{f_i^{n+1} - f_i^n}{\Delta t}$$

$$f_x = \frac{f_i^n - f_{i-1}^n}{\Delta x}$$

3. UPSTREAM DIFFERENCING METHOD

This method is also known as windward difference method. We use finite difference methods to solve the partition of wave equation.

$$f_t + c f_x = 0 \quad (1)$$

This is also called advection equation. With the help of this method, we use backward space difference provided that the wave speed c is positive and if c is negative, we have to ensure the stability by using forward difference. For $c > 0$ the above equation, may results as, at a grid point (i, n) discussed within the region shown in fig.1.

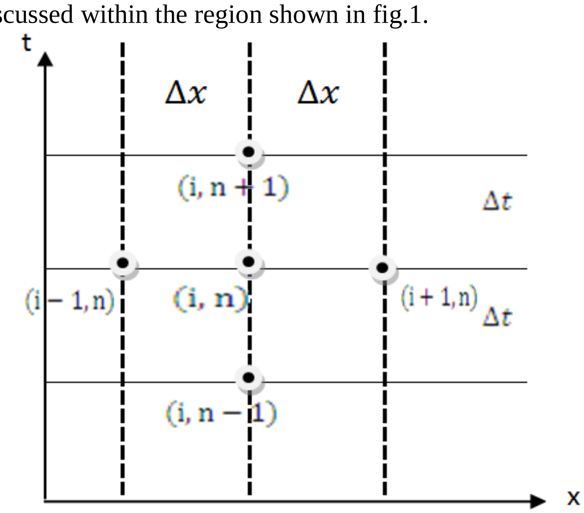

Figure-1 Grid points under discussion

For the solution of problems with this method, we need both boundary conditions and initial conditions. This method is also called an explicit method

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + c\,\frac{f_i^n - f_{i-1}^n}{\Delta x} = 0 \quad (2)$$

The above Equation can be solved explicitly for f_i^{n+1}.

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = -c\,\frac{f_i^n - f_{i-1}^n}{\Delta x}$$

$$f_i^{n+1} = f_i^n - c\,\frac{\Delta t}{\Delta x}\left(f_i^n - f_{i-1}^n\right) \quad (3)$$

Equation (3) is the upwind approximation of the equation (1) in the case c > 0.

4. MDE FOR UPWIND METHODS

The procedure of deriving the modified equation (ME) for upwind difference approximation [5, 6] is very alike to computing the local truncation error. We are using the difference approximations at a grid point(i,n).

We have from above equation.

$$\frac{{f_i}^{n+1}-{f_i}^{n}}{\Delta t}+C\left(\frac{{f_i}^{n}-f_{i-1}^{n}}{\Delta x}\right)=0 \qquad (4)$$

For obtaining a better form, we have to eliminate higher order time derivatives and the mixed time and space derivatives and higher order time derivations from equation (4). This can be done by repeated differentiation.

$$f_t + cf_x = \left(\frac{c}{2}\Delta x - \frac{c^2}{2}\Delta t\right) f_{xx} - \left(\frac{c^3}{3}\Delta t^2 - \frac{c^2}{2}\Delta t + \frac{c}{6}\Delta x^2\right) f_{xxx}\ldots \quad (5)$$

This is a modified equation for $f_t + cf_x = 0$ in the case c > 0. In this case we come to know that the exact solution to the advection equation is recovered by the upwind method.

5. Use of Modified Equation to examine the stability of upwind scheme

The right side of the modified the equation (5) is the truncation error since it represents the difference between the original PDE and the finite-difference approximation to it.

The lowest-order term of the truncation error in the present case contains the partial derivative f_{xx}, which makes this term similar to the viscous terms in the one dimensional Navier-Stokes equation. This may be written as:

$$\frac{\partial}{\partial x}(\tau_{xx}) = \frac{4}{3}\mu f_{xx} \qquad (6)$$

If a constant coefficient of viscosity is assumed thus, if $V = c\,\Delta t/\Delta x \neq 1$, the upwind differencing scheme brings an "artificial viscosity" into the solution. This is often called implicit artificial viscosity, other than the explicit artificial viscosity which is purposely added to a difference scheme. Artificial viscosity tends to reduce all gradients in the solution whether physically correct or numerically induced. This effect, which is the direct result of even derivative terms in the truncation error, is called dissipation.

Another quasi-physical effect of numerical schemes is called dispersion. This is the direct result of the odd derivative terms which appear in the truncation error. As a result of dispersion, phase relations between various waves are distorted. The combined effect of dissipation and dispersion is sometimes referred to as diffusion.

The modified equation (5) can be written as:-

$$f_t + cf_x = \sum_{n=1}^{\infty}\left(C_{2n}\frac{\partial^{2n} f}{\partial x^{2n}} + C_{2n+1}\frac{\partial^{2n+1} f}{\partial x^{2n+1}}\right) \qquad (7)$$

WhereC_{2n} and C_{2n+1}represent the coefficients of the even and odd spatial derivative terms respectively. Warming and Hyett [2] have shown that in the small wave number limit where all higher order terms are negligible, the necessary condition for stability is

$$(-1)^{l+1}C(2l) > 0 \qquad (8)$$

Where $C(2l)$ represent the coefficient of the lowest-order even derivative term in the right-hand side of a modified equation. They also showed that a more complete stability analysis leads to the following general necessary and sufficient condition.

$$\left(8\frac{\Delta t}{\Delta x^4}\right)\left[\frac{\Delta x^2}{3}C(2l) - \Delta t C^2(2l) - C(4l)\right] > 0 \quad if \qquad (9)$$

$$\left((-4)^{l+1}\frac{\Delta t}{2\Delta x^{2l+2}}\right)\left[\left(\frac{1}{12}\right)(3+l)\Delta x^2 C(2l) - C(2l+2)\right] > 0, \qquad (10)$$

$if\ l \geq 2$

The lowest-order even derivative coefficient in the right-hand side of the modified equation (5) is $C(2=2l)$, so $l=1$. From the inequality (8)

If

$$v = c\,\Delta t/\Delta x$$

then a necessary condition for stability is $C(2) > 0$, i.e.

$$\frac{c}{2}\Delta x\left(1-\frac{c\Delta t}{\Delta x}\right) > 0$$

$$\frac{c\Delta x}{2}(1-v) > 0$$

For $v < 1$, which is obtained from the Von Neumann stability analysis.

Warming and Hyett have also shown that the relative phase error for difference schemes applied to the equation

$f_t + cf_x = 0$, is

$$\frac{\phi}{\phi_c} = 1 - \frac{1}{c}\sum_{n=1}^{\infty}(-1)^n (k)^{2n} C_{2n+1} \qquad (11)$$

For small wave numbers, we need only to retain the lowest-order term. For the upwind differencing scheme $\beta = k\Delta x$ and $c > 0$, we find that

$$\frac{\phi}{\phi_c} = 1 - \frac{1}{c}(-1)\left(\frac{\beta}{\Delta x}\right)^2 C_3 = 1 - \frac{1}{6}(1 - 3v + 2v^2)\beta^2 \qquad (12)$$

This is Von Neuman stability analysis. Thus we have demonstrated that the Von Neumann stability analysis and the stability theory based on the modified equation are directly related.

6. EXAMINE THE STABILITY FOR UPWIND SCHEME

First we will discuss stability analysis for the upwind difference approximation of the following equation:

$$f_t + cf_x = 0 \qquad (13)$$

The upwind scheme of above equation is given by the following equation:

$$f_i^{n+1} = f_i^n - \upsilon\left(f_i^n - f_{i-1}^n\right) \qquad (14)$$

where $\upsilon = \frac{c\Delta t}{\Delta x}$ In the Von Neumann method, the independent solution of the difference equations are all of the form.

$$f_i^n = G^n\ e^{lkx} \qquad (15)$$

where, $I = \sqrt{-1}, k$ is a real wave number, and G= G(k) called the amplification factor, is in general a complex constant. The difference equations are stable if

$$|G| \leq 1$$

To find G, we substitute equation (15) in equation (14), and using.

$$f_i^{n+1} = G^{n+1}\ e^{lkx}$$

$$f_{i-1}^{n+1} = G^n\ e^{lk(x-\Delta x)}$$

We get

$$G^{n+1}\, e^{lkx} - \upsilon(G^n\, e^{lkx} - G^n\, e^{lk(x-\Delta x)}$$

$$G^{n+1} e^{IKx} = G^n e^{IKx} - v(\mathrm{G}^n\, e^{IKx} - G^n e^{IK(x-\Delta x)})$$

$$G^n.G.e^{IKx} = G^n.e^{IKx}[1 - v(1 - \overline{e}^{IK\Delta x})]$$

$$G = [1 - v(1 - \mathrm{e}^{IK\Delta x})]$$

$$= [1 - v(\mathrm{Cos K}\,\Delta\,\mathrm{x} - I Sin\,\mathrm{K}\,\Delta\,\mathrm{x})]$$

$$G = (1 - \mathrm{v} + Cos\beta) + I(-vSin\beta) \quad (16)$$

Where

$$\beta = K\Delta x$$

7. RESULTS AND DISCUSSION

The modulus of the amplification factor

$$|G| = [(1 - \mathrm{v} + Cos\beta)^2 + (-vISin\beta)^2]^{1/2} \quad (17)$$

is plotted in figure 2 for several values of v. The solid line is the graph of |G| for v = 0.5. The diamond signs are the graphs of |G| for v=0.75. The dashed lines is the graph of |G| for v = 1.0 and the plus signs are the graph of |G| for v=1.25. It is clear from this plot that v must be less than or equal to 1 if the Von Neumann stability condition $|G| \leq 1$ is to be met. The amplification factor, equation (16), can also be expressed in the exponential form for a complex number.

$$G = |G|\, e^{I\phi}$$

where ϕ is the phase angle given by:

$$\phi = \tan^{-1}\frac{\mathrm{Im(G)}}{\mathrm{Re(G)}} = \tan^{-1}\left[\frac{-v\sin\beta}{1 - v + v\cos\beta}\right]$$

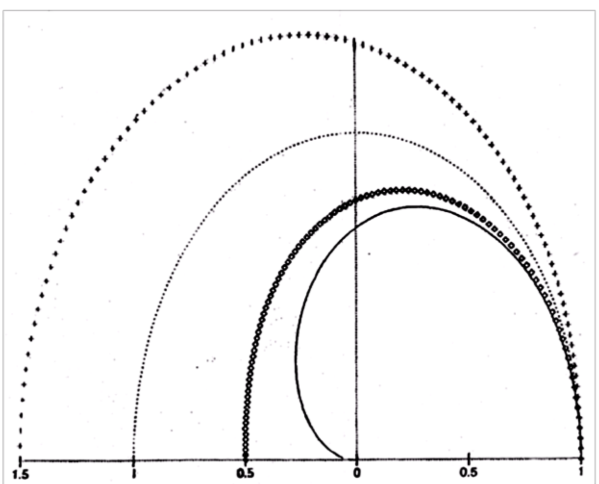

Figure-2 Amplification factor modulus for upwind scheme.

The phase angle for the exact solution of the convection equation (ϕ_c) is determined in a similar manner once the amplification factor is known. In order to find the exact amplification factor, we substitute the element solution

$$f = e^{at} e^{Ikx}$$

Into the equation $(f_t + cf_x = 0)$ and find that a = -Ikc, which gives

$$f = e^{Ik(x-ct)}$$

The exact amplification factor is then

$$G_c = \frac{f(\mathrm{t} + \Delta\,\mathrm{t})}{f(\mathrm{t})} = \frac{e^{Ik[x - c(t+\Delta t)]}}{e^{Ik(x-ct)}}$$

which reduces to

$$G_e = e^{-Ikc\Delta t} = e^{I\phi e}$$

where

$$\phi_c = -kc\Delta t = -\beta v$$

and

$$|G_e| = |e^{I\phi e}| = \left[(Cos\beta v)^2 + (-Sin\beta v)^2\right]^{1/2} = 1$$

The relative phase shift error after one time step is given by

$$\frac{\phi}{\phi_c} = \frac{\tan^{-1}[(-\mathrm{v}\sin\beta)/(1 - \mathrm{v} + v\cos\beta)]}{-\beta v}$$

and is plotted in Figure 3 for several values of v.

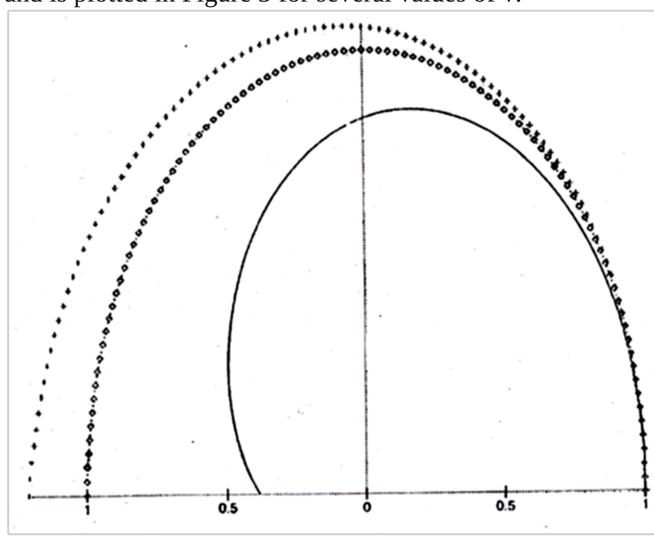

Figure-3 Relative phase error of upwind scheme

The solid line is the graph of ϕ/ϕ_c for v = 0.25. The diamond signs are the graphs of ϕ/ϕ_c for v=0.5, and the plus signs are the graphs of ϕ/ϕ_c for v = 0.75. For small wave numbers (i.e., small β), the relative phase error reduces to

$$\frac{\phi}{\phi_c} = 1 - \frac{1}{6}(1 - 3\mathrm{v} + 2\mathrm{v}^2)\beta^2$$

If the relative phase error exceeds 1 for a given value of β, the corresponding Fourier component of the numerical solution has a wave speed greater than the exact solution. This is a leading phase error. If the relative phase error is less than 1, the wave speed of the numerical solution is less than the exact wave speed. This is lagging phase error. The upwind differencing method has a leading phase error for 0.5<v<1 and a lagging phase error for v<0.5.

8. CONCLUSION

MDE's in specific problems are more convenient for discussing the solution behavior, including physical interpretation, i.e. accuracy, stability and consistency. Many ordinary and higher order boundary value problems have been analyzed with the help of modified equation. The appropriate solution converges rapidly to accurate solution. So we say that MDE's are more beneficial for future use.

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