APPLICATIONS OF *K*-WEYL FRACTIONAL INTEGRAL

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ABSTRACT: *In this paper, we prove some results of k - Weyl fractional integral. Then, we give proofs of some inequalities for k -Weyl fractional integral. When $k \to 1$, these results and inequalities hold good for the usual Weyl fractional integral.*

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KEYWORDS: k -Weyl fractional integral, k -gamma function, k -beta function, Chebyshev inequalities

1. INTRODUCTION

Diaz and Pariguan [1] paved the way for extensions of fractional calculus when they introduced the k -gamma and k -beta functions and the Pochhammer k -symbol(the generalized form of the classical Gamma function, beta function and the classical Pochhammer symbol).

Mubeen and Habibullah [2] have given the idea of k -fractional integrals. They defined the k -Riemann-Liouville fractional integral by using the k -Gamma function.

Diaz and Pariguan [1] have defined the k -gamma function as

$$\Gamma_k(x) = \int_0^\infty e^{\frac{t^k}{k}} t^{x-1} dt, \ \ \mathrm{Re}(x) > 0 . \qquad (1)$$

They also have defined the k -beta function as

$$B_k(x,y) = \frac{1}{k}\int_0^1 (1-t)^{\frac{y}{k}-1} t^{\frac{x}{k}-1)} dt \qquad (2)$$

$\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0$,

It is easy to see

$$\Gamma(x) = \lim_{k\to 1}\Gamma_k(x), \Gamma_k(x+k) = x\Gamma_k(x),$$

and

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}.$$

For any real number $\alpha \in (0,1)$ and k >0, Romero and Luque [3] have defined the k -Weyl fractional integral as

$$W_k^{\alpha}(f(x)) = \frac{1}{k\Gamma_k(\alpha)}\int_x^\infty (t-x)^{\frac{\alpha}{k}-1} f(t)dt, x \ge 0, t > 0. \qquad (3).$$

where, $\Gamma_k(\alpha)$ is the k -Gamma Euler function.

2. MAIN RESULTS

1.1 Theorem: Let f be continuous on $[0,\infty)$ and let $\alpha, \beta \in (0,1), k > 0.$ Then for $x \ge 0$

$$W_k^{\alpha}\left[W_k^{\beta}(f(x))\right] = \left[W_k^{\alpha+\beta}(f(x))\right] = W_k^{\beta}\left[W_k^{\alpha}(f(x))\right] \qquad (4).$$

Proof: Using (3), we get

$$W_k^{\alpha}\left[W_k^{\beta}(f(x))\right] = \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_x^\infty (t-x)^{\frac{\alpha}{k}-1}\left[\int_t^\infty (u-t)^{\frac{\beta}{k}-1} f(u)du\right]dt.$$

Using Fubini's theorem

$$W_k^{\alpha}\left[W_k^{\beta}(f(x))\right] = \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_x^\infty f(u)\left[\int_x^u (t-x)^{\frac{\alpha}{k}-1}(u-t)^{\frac{\beta}{k}-1} dt\right]du.$$

By substituting $y = \frac{u-t}{u-x}$

$$W_k^{\alpha}\left[W_k^{\beta}(f(x))\right] = \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_x^\infty (u-x)^{\frac{\alpha}{k}+\frac{\beta}{k}-1} f(u)\left[\int_0^1 (1-y)^{\frac{\alpha}{k}-1}(y)^{\frac{\beta}{k}-1} dy\right]du.$$

Using (2)

$$W_k^{\alpha}\left[W_k^{\beta}(f(x))\right] = \frac{1}{k\Gamma_k(\alpha+\beta)}\int_x^\infty (u-x)^{\frac{\alpha}{k}+\frac{\beta}{k}-1} f(u)du = \left[W_k^{\alpha+\beta}(f(x))\right].$$

Similarly, we can prove

$W_k^{\beta}\left[W_k^{\alpha}(f(x))\right] = \left[W_k^{\alpha+\beta}(f(x))\right]$ and get (4).

Also see Romero and Luque [2].

Example 1. Let α be a real numbers and $\alpha\in(0,1), k>0.$ Then for all $\mu>0$

$$W_k^{\alpha}[(e^{-\mu x}]=\frac{e^{-\mu x}}{(\mu k)^{\frac{\alpha}{k}}}.$$

Solution: Using (3), we get

$$W_k^{\alpha}[(e^{-\mu x}]=\frac{1}{k\Gamma_k(\alpha)}\int_x^{\infty}(t-x)^{\frac{\alpha}{k}-1}e^{-\mu t}dt.$$

By substituting $t-x=z$

$$W_k^{\alpha}[(e^{-\mu x}]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty}z^{\frac{\alpha}{k}-1}e^{-\mu(x+z)}dz.$$

By substituting $u=\mu z$

$$W_k^{\alpha}[(e^{-\mu x}]=\frac{e^{-\mu x}}{\mu^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_0^{\infty}u^{\frac{\alpha}{k}-1}e^{-u}du.$$

Using (1), we get the required result.

We now prove some inequalities involving the k-Weyl fractional integrals.

1.2 Theorem. Let f,g be two synchronous on $[0,\infty)$, $\alpha,\beta\in(0,1), k>0.$ Then for all $t>0$, the following inequalities for k –Weyl fractional integrals hold:

$$W_k^{\alpha}(1)W_k^{\alpha}\left(fg(t)\right)\geq W_k^{\alpha}\left(f(t)\right)W_k^{\alpha}\left(g(t)\right). \quad (5).$$

$$W_k^{\alpha}\left(fg(t)\right)W_k^{\beta}(1)+W_k^{\beta}\left(fg(t)\right)W_k^{\alpha}(1)$$
$$\geq W_k^{\alpha}\left(f(t)\right)W_k^{\beta}\left(g(t)\right)+W_k^{\alpha}\left(g(t)\right)W_k^{\beta}\left(f(t)\right). \quad (6)$$

Proof: Since the functions f,g are synchronous on $[0,\infty)$, then for all $x,y\geq 0$, we have $(f(x)-f(y))(g(x)-g(y))\geq 0.$

Therefore,

$$f(x)g(x)+f(y)g(y)\geq f(x)g(y)+f(y)g(x).$$

Multiplying both sides by $\frac{1}{k\Gamma_k(\alpha)}(x-t)^{\frac{\alpha}{k}-1}$ then integrating w.r.t. x over (t,∞)

$$\frac{1}{k\Gamma_k(\alpha)}\int_t^{\infty}(x-t)^{\frac{\alpha}{k}-1}f(x)g(x)dx+\frac{1}{k\Gamma_k(\alpha)}\int_t^{\infty}(x-t)^{\frac{\alpha}{k}-1}f(y)g(y)dx$$
$$\geq\frac{1}{k\Gamma_k(\alpha)}\int_t^{\infty}(x-t)^{\frac{\alpha}{k}-1}f(x)g(y)dx+\frac{1}{k\Gamma_k(\alpha)}\int_t^{\infty}(x-t)^{\frac{\alpha}{k}-1}f(y)g(x)dx.$$

Using (3), we get

$$W_k^{\alpha}\left(fg(t)\right)+f(y)g(y)W_k^{\alpha}(1)$$
$$\geq g(y)W_k^{\alpha}\left(f(t)\right)+f(y)W_k^{\alpha}\left(g(t)\right).$$

Again multiplying both sides by $\frac{1}{k\Gamma_k(\alpha)}(y-t)^{\frac{\alpha}{k}-1}$, then integrating w.r.t. y over (t,∞)

$$\frac{1}{k\Gamma_k(\alpha)}\int_t^{\infty}(y-t)^{\frac{\alpha}{k}-1}W_k^{\alpha}\left(fg(t)\right)dy+\frac{1}{k\Gamma_k(\alpha)}\int_t^{\infty}(y-t)^{\frac{\alpha}{k}-1}f(y)g(y)W_k^{\alpha}(1)dy$$
$$\geq\frac{1}{k\Gamma_k(\alpha)}\int_t^{\infty}(y-t)^{\frac{\alpha}{k}-1}g(y)W_k^{\alpha}\left(f(t)\right)dy+\frac{1}{k\Gamma_k(\alpha)}\int_t^{\infty}(y-t)^{\frac{\alpha}{k}-1}f(y)W_k^{\alpha}\left(g(t)\right)dy.$$

Using (3), we get (5).

Multiplying both sides of (8) by $\frac{1}{k\Gamma_k(\beta)}(y-t)^{\frac{\beta}{k}-1}$, then integrating w.r.to y over (t,∞)

$$\frac{1}{k\Gamma_k(\beta)}\int_t^{\infty}(y-t)^{\frac{\beta}{k}-1}W_k^{\alpha}\left(fg(t)\right)dy+\frac{1}{k\Gamma_k(\beta)}\int_t^{\infty}(y-t)^{\frac{\beta}{k}-1}f(y)g(y)W_k^{\alpha}(1)dy$$
$$\geq\frac{1}{k\Gamma_k(\beta)}\int_t^{\infty}(y-t)^{\frac{\beta}{k}-1}g(y)W_k^{\alpha}\left(f(t)\right)dy+\frac{1}{k\Gamma_k(\beta)}\int_t^{\infty}(y-t)^{\frac{\beta}{k}-1}f(y)W_k^{\alpha}\left(g(t)\right)dy.$$

Using (3), we obtain (6).

1.3 Theorem. Let f,g are two synchronous on $[0,\infty)$ and h be a function such that $h:[0,\infty)\rightarrow[0,\infty)$, $\alpha,\beta\in(0,1), k>0.$ Then for $t>0$

$$W_k^{\alpha}fgh(t)W_k^{\beta}(1)+W_k^{\alpha}(1)W_k^{\beta}fgh(t)$$
$$\geq W_k^{\alpha}fh(t)W_k^{\beta}g(t)+W_k^{\alpha}gh(t)W_k^{\beta}f(t)-W_k^{\alpha}h(t)W_k^{\beta}fg(t)$$
$$-W_k^{\alpha}fg(t)W_k^{\beta}h(t)+W_k^{\alpha}f(t)W_k^{\beta}gh(t)+W_k^{\alpha}g(t)W_k^{\beta}fh(t). \quad (7).$$

Proof: Since *f*, *g* are two synchronous on $[0,\infty)$, then for all $x,y\geq 0$,

$$(f(x)-f(y))(g(x)-g(y))(h(x)+h(y))\geq 0.$$

By opening the above, we get

$$f(x)g(x)h(x)+f(y)g(y)h(y)$$
$$\geq f(x)g(y)h(x)+f(y)g(x)h(x)-f(y)g(y)h(x)$$
$$-f(x)g(x)h(y)+f(x)g(y)h(y)+f(y)g(x)h(y).$$

Multiplying both sides by $\frac{1}{k\Gamma_k(\alpha)}(x-t)^{\frac{\alpha}{k}-1}$, then integrating w.r.t. x over (t,∞)

$$\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} f(x)g(x)h(x)dx + f(y)g(y)h(y)\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1}dx$$

$$\geq g(y)\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} f(x)h(x)dx + f(y)\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} g(x)h(x)dx$$

$$-f(y)g(y)\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} h(x)dx - h(y)\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} f(x)g(x)dx$$

$$+g(y)h(y)\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} f(x)dx + f(y)h(y)\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} g(x)dx.$$

Using (3), we get

$$W_k^\alpha fgh(t) + f(y)g(y)h(y)W_k^\alpha(1)$$
$$\geq g(y)W_k^\alpha fh(t) + f(y)W_k^\alpha gh(t) - f(y)g(y)W_k^\alpha h(t)$$
$$-h(y)W_k^\alpha fg(t) + g(y)h(y)W_k^\alpha f(t) + f(y)h(y)W_k^\alpha g(t).$$

Multiplying both sides by $\frac{1}{k\Gamma_k(\beta)}(y-t)^{\frac{\beta}{k}-1}$, then integrating w.r.t. y over (t,∞), we obtain

$$W_k^\alpha fgh(t)\frac{1}{k\Gamma_k(\beta)}\int_t^\infty (y-t)^{\frac{\beta}{k}-1}dy + W_k^\alpha(1)\frac{1}{k\Gamma_k(\beta)}\int_t^\infty (y-t)^{\frac{\beta}{k}-1} f(y)g(y)h(y)dy$$

$$\geq W_k^\alpha fh(t)\frac{1}{k\Gamma_k(\beta)}\int_t^\infty (y-t)^{\frac{\beta}{k}-1} g(y)dy + W_k^\alpha gh(t)\frac{1}{k\Gamma_k(\beta)}\int_t^\infty (y-t)^{\frac{\beta}{k}-1} f(y)dy$$
$$-W_k^\alpha h(t)\frac{1}{k\Gamma_k(\beta)}\int_t^\infty (y-t)^{\frac{\beta}{k}-1} f(y)g(y)dy - W_k^\alpha fg(t)\frac{1}{k\Gamma_k(\beta)}\int_t^\infty (y-t)^{\frac{\beta}{k}-1} h(y)dy$$
$$+W_k^\alpha f(t)\frac{1}{k\Gamma_k(\beta)}\int_t^\infty (y-t)^{\frac{\beta}{k}-1} g(y)h(y)dy + W_k^\alpha g(t)\frac{1}{k\Gamma_k(\beta)}\int_t^\infty (y-t)^{\frac{\beta}{k}-1} f(y)h(y)dy.$$

Which leads to (7) by the use of (3).

Corollary:1 Let *f*, *g* are two synchronous on $[0,\infty)$, $h \geq 0$, $\alpha \in (0,1), k > 0.$ Then for $t > 0$

$$W_k^\alpha fgh(t)W_k^\alpha(1) \geq$$
$$W_k^\alpha fh(t)W_k^\alpha g(t) + W_k^\alpha gh(t)W_k^\alpha f(t) - W_k^\alpha h(t)W_k^\alpha fg(t).$$

1.4 Theorem. Let *f*, *g* are two synchronous on $[0,\infty)$, $h \geq 0$, $\alpha,\beta \in (0,1), k > 0.$ Then for $t > 0$

$$W_k^\alpha fgh(t)W_k^\beta(1) - W_k^\alpha(1)W_k^\beta fgh(t)$$
$$\geq W_k^\alpha fh(t)W_k^\beta g(t) + W_k^\alpha gh(t)W_k^\beta f(t) - W_k^\alpha h(t)W_k^\beta fg(t)$$
$$+W_k^\alpha fg(t)W_k^\beta h(t) - W_k^\alpha f(t)W_k^\beta gh(t) - W_k^\alpha g(t)W_k^\beta fh(t).$$

Proof: This can be proved by the method used in Theorem 1.3.

1.5 Theorem. Let *f*, *g* are two synchronous on $[0,\infty)$, $\alpha,\beta \in (0,1), k > 0.$ Then for $t > 0$

$$W_k^\alpha\left(f^2(t)\right)W_k^\beta(1) + W_k^\beta\left(g^2(t)\right)W_k^\alpha(1) \geq 2W_k^\alpha\left(f(t)\right)W_k^\beta\left(g(t)\right). \quad (8).$$

$$W_k^\alpha f^2(t)W_k^\beta g^2(t) + W_k^\beta f^2(t)W_k^\alpha g^2(t) \geq 2W_k^\alpha fg(t)W_k^\beta fg(t). \quad (9).$$

Proof: Since *f*, *g* are two synchronous on $[0,\infty)$, then for all $x, y \geq 0$, we have

$$(f(x) - g(y))^2 \geq 0.$$

Then, we get $f^2(x) + g^2(y) \geq 2f(x)g(y)$.

Multiplying both sides by $\frac{1}{k\Gamma_k(\alpha)}(x-t)^{\frac{\alpha}{k}-1}$, then integrating w.r.to x over (t,∞), we get

$$\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} f^2(x)dx + \frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} g^2(y)dx$$

$$\geq \frac{2}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} f(x)g(y)dx.$$

Using (3), we obtain

$$W_k^\alpha\left(f^2(t)\right) + g^2(y)W_k^\alpha(1) \geq 2g(y)W_k^\alpha\left(f(t)\right).$$

Multiplying both sides by $\frac{1}{k\Gamma_k(\beta)}(y-t)^{\frac{\beta}{k}-1}$, then integrating w.r.to y over (t,∞), we get

$$\frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} f^2(x)dx + \frac{1}{k\Gamma_k(\alpha)}\int_t^\infty (x-t)^{\frac{\alpha}{k}-1} g^2(y)dx$$

$$\geq \frac{2}{k\Gamma_k(\beta)}\int_t^\infty (y-t)^{\frac{\beta}{k}-1} g(y)W_k^\alpha\left(f(t)\right)dy.$$

Using (3), we get (8).

Since $[f(x)g(y) - f(y)g(x)]^2 \geq 0$.

By expansion and using the above method we obtain (9).

Corollary:2 Let f, g be two synchronous on $[0,\infty)$, then for all $t > 0$, $\alpha \in (0,1), k > 0.$ We can get

$$W_k^\alpha f(1)[W_k^\alpha f^2(t)+W_k^\alpha g^2(t)] \geq 2W_k^\alpha f(t) W_k^\alpha g(t).$$

and

$$W_k^\alpha f^2(t) W_k^\alpha g^2(t) \geq [W_k^\alpha fg(t)]^2.$$

1.8 Theorem. Let $f:\mathbb{R}\to\mathbb{R}$ and defined by

$$\overline{f}(x)=\int_x^\infty f(u)du$$

Then for $\alpha\in(0,1), k>0,$

$$W_k^\alpha \overline{f}(x) = kW_k^{\alpha+k} f(x). \qquad (10).$$

Proof: Using (3) and substituting the value of $\overline{f}(x)$

$$W_k^\alpha \overline{f}(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^\infty (t-x)^{\frac{\alpha}{k}-1}[\int_x^\infty f(u)du]dt.$$

Using Fubini's theorem

$$W_k^\alpha \overline{f}(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^\infty f(u)[\int_x^u (t-x)^{\frac{\alpha}{k}-1}dt]du.$$

Integrating and using (3), we obtain (10).

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