ON CONVERGENCE OF IMPLICIT ITERATION SCHEME FOR TWO HEMICONTRACTIVE MAPPINGS

Arif Rafiq
Hajvery University, 43-52 Industrial Area, Gulberg-III, Lahore, Pakistan aarafiq@gmail.com

ABSTRACT The purpose of this paper is to characterize conditions for the convergence of the implicit iterative scheme to the common fixed point of two ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space.

Key Words Implicit iterative scheme, ϕ-hemicontractive mappings, Banach spaces

INTRODUTION

Let K be a nonempty subset of an arbitrary Banach space E and E^* be its dual space. The symbols $D(T)$, $R(T)$ and $F(T)$ stand for the domain, the range and the set of fixed points of T (for a single-valued map $T: X \to X$, $x \in X$ is called a fixed point of T iff $T(x) = x$). We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \}.$$

Let $T : D(T) \subseteq X \to X$ be an operator.

Definition 1 [1, 4, 10, 19]

(i) T is said to be strongly pseudocontractive if there exists a $t > 1$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re\langle Tx - Ty, j(x-y)\rangle \le \frac{1}{t}\|x-y\|^2.$$

(ii) T is said to be strictly hemicontractive if $F(T) \ne \varphi$ and if there exists a $t > 1$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re\langle Tx - q, j(x-q)\rangle \le \frac{1}{t}\|x-q\|^2.$$

(iii) T is said to be ϕ-strongly pseudocontractive if there exists a strictly increasing function $\phi : [0,\infty) \to 0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

(iv) T is said to be ϕ-hemicontractive if $F(T) \ne \varphi$ and if there exists a strictly increasing function $\phi : [0,\infty) \to 0,\infty)$ with $\phi(0) = 0$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re\langle Tx - q, j(x-q)\rangle \le \|x-q\|^2 - \phi(\|x-q\|)\|x-q\|.$$

Clearly, each strictly hemicontractive operator is ϕ-hemicontractive .

Chidume [1] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudo-contractive mapping from a bounded closed convex subset of L_p $(or\ l_p)$ into itself. Afterwards, several authors generalized this result of Chidume in various directions [4, 6-9, 15-16, 18-19].

In 2001, Xu and Ori [20] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1,2,\ldots,N\}$), with $\{\alpha_n\}$ a real sequence in $(0,1)$, and an initial point $x_0 \in K$:

$$x_1 = (1-\alpha_1)x_0 + \alpha_1 T_1 x_1,$$
$$x_2 = (1-\alpha_2)x_1 + \alpha_2 T_2 x_2,$$
$$\vdots$$
$$x_N = (1-\alpha_N)x_{N-1} + \alpha_N T_N x_N,$$
$$x_{N+1} = (1-\alpha_{N+1})x_N + \alpha_{N+1} T_{N+1} x_{N+1},$$
$$\vdots$$

which can be written in the following compact form:

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_n x_n, \text{for all} n \ge 1, \qquad \text{(XO)}$$

where $T_n = T_{n(mod\, N)}$ (here the $mod\, N$ function takes values in I). Xu and Ori [20] proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [11], Oslilike proved the following results.

Theorem 2 Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N strictly pseudocontractive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \ne \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:

(i) $0 < \alpha_n < 1,$

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty,$

(iii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty.$

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (XO). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if $\lim_{n\to\infty}\inf d(x_n, F) = 0$.

Remark 3 One can easily see that for $\alpha_n = 1 - \frac{1}{n^{\frac{1}{2}}}$, $\sum(1-\alpha_n)^2 = \infty$. Hence the results of Osilike [11] are needed to be improve.

The purpose of this paper is to characterize conditions for the convergence of the implicit iterative scheme to the common fixed point of two ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space. Our results extend and improve most results in recent literature [1-4, 6-9, 11-13, 15-16, 18-19].

Preliminaries

The following results are now well known.

Lemma 4 [17] For all $x, y \in X$ and $j(x+y) \in J(x+y)$,

$$\|x+y\|^2 \le \|x\|^2 + 2Re\langle y, j(x+y)\rangle.$$

Lemma 5 Let $\{\theta_n\}$ be a sequence of nonnegative real numbers, $\{\lambda_n\}$ be a real sequence satisfying

$$0 \le \lambda_n \le 1, \sum_{n=0}^{\infty}\lambda_n = \infty.$$

Suppose there exists a strictly increasing function $\phi:[0,\infty) \to 0,\infty)$ with $\phi(0) = 0$. If there exists a positive integer n_0 such that

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \phi(\theta_{n+1})\theta_{n+1} + \sigma_n + \gamma_n,$$

for all $n \ge n_0$, with $\sigma_n \ge 0$, $\forall n \in \mathrm{N}$, $\sigma_n = 0(\lambda_n)$ and $\sum_{n=0}^{\infty}\gamma_n < \infty$, then $\lim_{n\to\infty}\theta_n = 0$.

Main Results

Now we prove our main results.

Theorem 6 Let K be a nonempty convex subset of an arbitrary Banach space X and let T_i; $i = 1,2$ be two uniformly continuous and ϕ-hemicontractive mappings. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n^i\}_{n=1}^{\infty}$, $i = 1,2$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{2}\beta_n^i + \alpha_n = 1$ and satisfying conditions (i) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, and (ii) $\lim_{n\to\infty}(1-\alpha_n) = 0$.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by

$$x_n = \alpha_n x_{n-1} + \sum_{n=1}^{2}\beta_n^i T_i x_n, n \ge 1. \tag{3.1}$$

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the common fixed point q of T_i; $i = 1,2$,

(b) $\lim_{n\to\infty} T_i x_n = q; i = 1,2,$

(c) $\{T_i x_n\}_{n=1}^{\infty}$; $i = 1,2$ are bounded .

Proof Since each T_i; $i = 1,2$ is ϕ-hemicontractive, it follows that $\cap_{i=1}^{2} F(T_i)$ is a singleton. Let $\cap_{i=1}^{2} F(T_i) = \{q\}$ for some $q \in K$.

Suppose that $\lim_{n\to\infty} x_n = q$, then the uniform continuity of T_i; $i = 1,2$ yields that

$$\lim_{n\to\infty} T_i x_n = q; i = 1,2.$$

Therefore $\{T_i x_n\}_{n=1}^{\infty}$ are bounded; $i = 1,2$.

Put

$$M_1 = \|x_0 - q\| + \sum_{n=1}^{2}\sup_{n\ge1}\|T_i x_n - q\|. \tag{3.2}$$

It is clear that $\| x_0 - q \| \le M_1$. Let $\| x_{n-1} - q \| \le M_1$. Next we will prove that $\| x_n - q \| \le M_1$.

Consider

$$\|x_n - q\| = \left\|\alpha_n x_{n-1} + \sum_{n=1}^{2}\beta_n^i T_i x_n - q\right\|$$

$$= \left\|\alpha_n (x_{n-1} - q) + \sum_{n=1}^{2}\beta_n^i (T_i x_n - q)\right\|$$

$$\le \alpha_n \|x_{n-1} - q\| + \sum_{n=1}^{2}\beta_n^i \|T_i x_n - q\|$$

$$\le \alpha_n M_1 + \sum_{n=1}^{2}\beta_n^i \|T_i x_n - q\|$$

$$= \alpha_n\left(\|x_0 - q\| + \sum_{n=1}^{2}\sup_{n\ge1}\|T_i x_n - q\|\right) + \sum_{n=1}^{2}\beta_n^i \|T_i x_n - q\|$$

$$\le \|x_0 - q\| + \left(\alpha_n \sum_{n=1}^{2}\sup_{n\ge1}\|T_i x_n - q\| + \sum_{n=1}^{2}\beta_n^i \|T_i x_n - q\|\right)$$

$$\le \|x_0 - q\| + \left(\left(1 - \sum_{n=1}^{2}\beta_n^i\right)_{i=1}^{2} \sup_{n\ge1}\|T_i x_n - q\| + \sum_{n=1}^{2}\beta_n^i \sup_{n\ge1}\|T_i x_n - q\|\right)$$

$$= \|x_0 - q\| + \sum_{n=1}^{2}\sup_{n\ge1}\|T_i x_n - q\|$$

$$= M_1.$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n\ge1}$ is bounded. Thus there is a constant $M > 0$ satisfying

$$M = \max\{\sup_{n\geq 1}\{\|T_i x_n - q\|^2\}_{i=1,n\geq 1}^2\} + \sup_{n\geq 1}\|x_n - q\| < \infty. \quad (3.3)$$

By virtue of Lemma 4 and (3.1), we infer that

$$\|x_n - q\|^2 = \left\|\alpha_n x_{n-1} + \sum_{n=1}^{2}\beta_n^i T_i x_n - q\right\|^2$$

$$= \left\|\alpha_n (x_{n-1} - q) + \sum_{n=1}^{2}\beta_n^i (T_i x_n - q)\right\|^2$$

$$\leq \alpha_n^2\|x_{n-1} - q\|^2 + 2\sum_{n=1}^{2}\beta_n^i Re\langle T_i x_n - q, j(x_n - q)\rangle$$

$$\leq \alpha_n^2\|x_{n-1} - q\|^2 + 2\sum_{n=1}^{2}\beta_n^i\|x_n - q\|^2$$

$$-2\sum_{n=1}^{2}\beta_n^i\phi(\|x_n - q\|)\|x_n - q\|$$

$$= \alpha_n^2\|x_{n-1} - q\|^2 + 2(1-\alpha_n)\|x_n - q\|^2$$

$$-2(1-\alpha_n)\phi(\|x_n - q\|)\|x_n - q\|.$$

Consider

$$\|x_n - q\|^2 = \left\|\alpha_n x_{n-1} + \sum_{n=1}^{2}\beta_n^i T_i x_n - q\right\|^2$$

$$= \left\|\alpha_n (x_{n-1} - q) + \sum_{n=1}^{2}\beta_n^i (T_i x_n - q)\right\|^2$$

$$\leq \alpha_n\|x_{n-1} - q\|^2 + \sum_{n=1}^{2}\beta_n^i\|T_i x_n - q\|^2$$

$$\leq \alpha_n\|x_{n-1} - q\|^2 + \sum_{n=1}^{2}\beta_n^i \sup_{n\geq 1}\|T_i x_n - q\|^2$$

$$\leq \alpha_n\|x_{n-1} - q\|^2 + \sum_{n=1}^{2}\beta_n^i \max\{\sup_{n\geq 1}\{\|T_i x_n - q\|^2\}_{i=1,n\geq 1}^2\}$$

$$\leq \|x_{n-1} - q\|^2 + M(1-\alpha_n), \quad (3.5)$$

where the first inequality holds by the convexity of $\|.\|^2$.

Substituting (3.5) in (3.4), we get

$$\|x_n - q\|^2 \leq [\alpha_n^2 + 2(1-\alpha_n)]\|x_{n-1} - q\|^2 + 2M(1-\alpha_n)^2 \quad (3.6)$$

$$-2(1-\alpha_n)\phi(\|x_n - q\|)\|x_n - q\|$$

$$= \left(1 + (1-\alpha_n)^2\right)\|x_{n-1} - q\|^2 + 2M(1-\alpha_n)^2$$

$$-2(1-\alpha_n)\phi(\|x_n - q\|)\|x_n - q\|$$

$$\leq \|x_{n-1} - q\|^2 + M(M+2)(1-\alpha_n)^2$$

$$-2(1-\alpha_n)\phi(\|x_n - q\|)\|x_n - q\|$$

$$= \|x_{n-1} - q\|^2 + (1-\alpha_n)l_n - 2(1-\alpha_n)\phi$$

$$(\|x_n - q\|)\|x_n - q\|,$$

where

$$l_n = M(M+2)(1-\alpha_n) \to 0, \quad (3.7)$$

as $n \to \infty$. (3.4)

Denote

$$\theta_n = \| x_{n-1} - p \|,$$

$$\lambda_n = 2(1-\alpha_n),$$

$$\sigma_n = (1-\alpha_n)l_n.$$

Condition (i) assures the existence of a rank $n_0 \in \mathrm{N}$ such that $\lambda_n = 2(1-\alpha_n) \leq 1,$ for all $n \geq n_0.$ Now with the help of (ii), (3.7) and Lemma 5, we obtain from (3.6) that

$$\lim_{n\to\infty}\| x_n - p \| = 0,$$

completing the proof. .

Corrollary 7 Let K be a nonempty convex subset of an arbitrary Banach space X and let $T_i;\ i = 1,2$ be two Lipschitz and ϕ-hemicontractive mappings. Let $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty},\ i = 1,2$ be real sequences in $[0,1]$ such that $\sum_{n=1}^{2}\beta_n^i + \alpha_n = 1$ and satisfying conditions (i) $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty,$ and (ii) $\lim_{n\to\infty}(1-\alpha_n) = 0.$

Suppose that $\{x_n\}_{n=0}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by

$$x_n = \alpha_n x_{n-1} + \sum_{n=1}^{2}\beta_n^i T_i x_n, n \geq 1.$$

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the common fixed point q of $T_i;\ i = 1,2,$

(b) $\lim_{n\to\infty} T_i x_n = q; i = 1,2,$

(c) $\{T_i x_n\}_{n=1}^{\infty};\ i = 1,2$ are bounded .

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