GROWTH RATE OF BRAID MONOIDS $\mathbf{MB}_{n+1}, n \le 6$

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***ABSTRACT**: In [3] we proved that the growth rate of all the spherical Artin monoids is less than 4. In [11] we gave an algorithm to find the Hilbert series of the braid monoids $\mathbf{MB}_{n+1}$ and found the Hilbert series and the growth rate of $\mathbf{MB}_3$ and $\mathbf{MB}_4$, in particular. In [12] we gave the Hilbert series and the growth rates of $\mathbf{MB}_5$ and $\mathbf{MB}_6$. In this paper we compute the Hilbert series and the growth rate of $\mathbf{MB}_7$.*

Key words: irreducible words, Hilbert series, Growth rate.

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1.INTRODUCTION

The braid group $\mathbf{B}_{n+1}$ admits the following classical presentation given by Artin: [2]

$$\mathbf{B}_{n+1} = \left\langle x_1, x_2, \dots, x_n \;\middle|\; \begin{matrix} x_i x_j = x_j x_i \text{ if } |i-j| \ge 2 \\ x_{i+1} x_i x_{i+1} = x_i x_{i+1} x_i \text{ if } 1 \le i \le n-1 \end{matrix} \right\rangle.$$

Elements of $\mathbf{B}_{n+1}$ are words expressed in the generators $x_1, x_2, \dots, x_n$ and their inverses. The braid monoid $\mathbf{MB}_{n+1}$ has the similar presentation

$$\mathbf{MB}_{n+1} = \left\langle y_1, y_2, \dots, y_n \;\middle|\; \begin{matrix} y_i y_j = y_j y_i \text{ if } |i-j| \ge 2 \\ y_{i+1} y_i y_{i+1} = y_i y_{i+1} y_i \text{ if } 1 \le i \le n-1 \end{matrix} \right\rangle.$$

Garside [9] proved that the map $\psi : \mathbf{MB}_{n+1} \to \mathbf{B}_{n+1}$ given by $\psi(y_i) = x_i$ is injective. The elements of $\mathbf{MB}_{n+1}$ are called *positive braids*.

In 1972, P. Deligne [8] proved that the Hilbert series (will be defined later) of all the Artin monoids are rational functions.

In 1992, P. Xu [13] found the Hilbert series of the braid monoids $\mathbf{MB}_3$ and $\mathbf{MB}_4$ and she also proved that the Hilbert series of $\mathbf{MB}_{n+1}$ is a rational function. She developed a linear system for $\mathbf{MB}_{n+1}$ of size $n!$ and she succeeded to reduce it to $2^{n-1} + 2^{[\frac{n-1}{2}]} - 2$ equations.

In 2003, Bokut [bok] gave the non-commutative Gröbner bases or complete presentation of the braid monoid $\mathbf{MB}_{n+1}$ (with the length-lexicographic order induced by $x_1 < \cdots < x_n$) and proved:

Theorem 1 [5]. A complete presentation (Gröbner bases) of $\mathbf{MB}_{n+1}$ consists of the following relations:

(i) $x_s x_k = x_k x_s, s-k \ge 2,$

(ii) $x_{i+1} x_i \alpha(i-1,1) x_{i+1} x_i \dots x_j = x_i x_{i+1} x_i \alpha(i-1,1) x_i \dots x_j \sum \beta(i,j)$, $1 \le i \le n-1$, $1 \le j \le n+1$

(For notations see Section sec 2.) In [11] we slightly modified the complete presentation of $\mathbf{MB}_{n+1}$ (given by Bokut) to make it reduced (i.e., all the relations do not contain reducible words) for the purpose of computation of Hilbert series. Using the reduced complete presentation (non-commutative Gröbner bases) of $\mathbf{MB}_{n+1}$ we found another system of equations to compute the Hilbert series. We constructed a linear system of equations for reducible as well as for irreducible words. The size of the system is $n^2 + 2n - 3$ which is much smaller than the size $2^{n-1} + 2^{[\frac{n-1}{2}]} - 2$ of Xu's system for $n \ge 7$. Using this system we gave an algorithm to compute inductively the Hilbert series of $\mathbf{MB}_{n+1}$.

Definition 1 [1]. Let G be a finitely generated group and S be a finite set of generators of G. The *word length* $l_S(g)$ of an element $g \in G$ is the smallest integer n for which there exist $s_1, \dots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$.

Definition 2 [1]. Let G be a finitely generated group and S be a finite set of generators of G. The *growth function* of the pair (G, S) associates to an integer $k \ge 0$ the number $a(k)$ of elements $g \in G$ such that $l_S(g) = k$ and the corresponding *spherical growth series* or the *Hilbert series* is given by $P_G(t) = \sum_{k=0}^{\infty} a(k) t^k$.

For a sequence $\{s_k\}_{k \ge 1}$ of positive numbers, we define the growth rate by:

Definition 3. We say that $\{s_k\}_{k \ge 1}$ has a growth rate γ (γ is a positive real number) if

$$\overline{\lim_k} \exp\left(\frac{\log s_k}{k}\right) = \gamma.$$

In [4] we have proved that the growth rate of all the spherical

Artin monoids is less than 4. In [11] we showed that the growth functions are exponential and the growth rates are (approximately) 1.618 and 2.0868 respectively for the monoids $\mathbf{MB}_3$ and $\mathbf{MB}_4$.

In this paper we compute the Hilbert series of the braid monoids $\mathbf{MB}_5$ and $\mathbf{MB}_6$ and calculate the growth rates of the above monoids that are 2.395 and 2.6 respectively.

2. Some Necessary Notions

All the following notions are in [1,4-7] under different names: complete presentation, Gröbner bases, presentation with solvable ambiguities, rewriting system and so on. In the free monoid generated by $x_1,\dots,x_n$ the *total order* on the set of generators given by $x_1<\cdots<x_n$ is extended to *length-lexicographic order*. A relation $\mathbf{R}$ is written in the form $a_i=b_i$ where a_i is a monomial greater than b_i. We denote by $a_i(\mathbf{R})$ and $b_i(\mathbf{R})$ the terms a_i and b_i respectively of the given relation $\mathbf{R}$. In a monoid (group), a word containing the L.H.S. of a relation is called *reducible word* and a word which does not contain the L.H.S. of a relation is called *irreducible word*. We denote A by the set of irreducible words and by B the set of reducible words. Let us introduce some notations.

- We denote by $\alpha(i,j)=\alpha(x_i,x_{i-1},\dots,x_j), i\ge j$ an arbitrary irreducible word (possibly e $x_i,x_{i-1},\dots,x_j$ and $\alpha(i,i)=\alpha(x_i)$, a word in the generator x_i. We denote the ``shift" of α by $\Sigma\alpha(i,j)=\alpha(x_{i+1},\dots,x_{j+1})$.
- If $U=U_1W$, $V=WV_1$ are the given words, then we denote their *overlap* (at W) by $U\times_W V=U_1WV_1$.
- We will use $i^a j^b k^c\cdots$ for a word $x_i^a x_j^b x_k^c\cdots$(especially in overlapping words) when required.
- $\mathbf{U}_{*,\beta}=$ set of irreducible words ending with β and $\mathbf{U}_{\delta,*}=$ set of irreducible words starting with δ.
- Suppose $\beta=\alpha\gamma$ and $\delta=\gamma\varepsilon$; then $\mathbf{U}_{*,\beta}\times_\gamma \mathbf{V}_{\delta,*}=\{U\times_\gamma V \;:\; U\in\mathbf{U}_{*,\beta}, V\in\mathbf{V}_{\delta,*}\}$

3. Hilbert Series of $\mathbf{MB}_5$ and $\mathbf{MB}_6$

We are using the following notions: generally $A_\alpha^{[n+1]}$ and $B_{\alpha,\omega}^{[n+1]}$ be the irreducible and reducible words respectively in $\mathbf{MB}_{n+1}$, and α is related with the prefix (beginning) of a word and ω is related with the suffix (end) of the word. For example $A_{k(k-1)\cdots i}^{[n+1]}$ denotes the set of irreducible words in $\mathbf{MB}_{n+1}$ starting with $x_k x_{k-1}\cdots x_i$; $B_{j,k}^{[n+1]}$ denotes the set of reducible words starting with $x_n x_{n-1}\cdots x_j$ and ending with $x_n x_{n-1}\cdots x_k$. As special cases we use the following notations: if $j=*$ then the word will start with $x_n x_{n-1}$ and if $j=n-1$ then the word will start with $x_n x_{n-1}^2$. Also a special reducible word $x_k x_{k-1} x_k$ is denoted by $B_{\phi,k}^{[n+1]}$. All the above sets are graded by length, so we can introduce the Hilbert series of these sets. Let $Q_{\alpha,\omega}^{[n+1]}(t)$, $P_\alpha^{[n+1]}(t)$ and $\mathbf{H}_M^{[n+1]}(t)$ denote the Hilbert series of $B_{\alpha,\omega}^{[n+1]}$, $A_\alpha^{[n+1]}$ and $A^{[n+1]}$ respectively for the mo**noid** M **, where** $A^{[n+1]}=\{e\}\coprod A_1^{[n+1]}\coprod A_2^{[n+1]}\coprod\cdots\coprod A_n^{[n+1]}$. In [11] we proved Lemma 1, 2 and 3 (using the reduced complete presentation) and constructed a linear system for the reducible words in $\mathbf{MB}_{n+1}$.

Lemma 1 [11]. The following relations hold for the reducible words in $\mathbf{MB}_{n+1}$.

1) $Q_{n-1,1}^{[n+1]}=t^{n+2}P_{n-1}^{[n]}-\sum_{j=2}^{n-1}t^{j-1}Q_{n-1,j}^{[n+1]}$.
2) $Q_{n-2,n}^{[n+1]}=t^3 P_{n-2}^{[n-1]}$.
3) $Q_{n-2,n-1}^{[n+1]}=t^4 P_{n-2}^{[n-1]}P_1^{[2]}-t^2 Q_{*,n-1}^{[n]}P_1^{[2]}$.
4) $$Q_{n-2,i}^{[n+1]}=t^{n-i+3}P_{(n-2)\dots k}^{[n-1]}P_{n-i}^{[n-i+1]}-\sum_{j=i+1}^{n-1}t^{j-i}Q_{k,j}^{[n+1]}-\sum_{j=i}^{n-1}t^{j-i+2}Q_{k,j}^{[n]}P_{(n-i)\dots j-i+1}^{[n-i+1]},$$ $i=1,\dots,n-2.$

Lemma 2 [11]. For $k=1,\dots,n-3$ the following relations hold for the reducible words in $\mathbf{MB}_{n+1}$.

1) $Q_{k,n}^{[n+1]}=t^3 P_{(n-2)\cdots k}^{[n-1]}$.
2) $Q_{k,n-1}^{[n+1]}=t^4 P_{(n-2)\cdots k}^{[n-1]}P_1^{[2]}-t^2 Q_{k,n-1}^{[n]}P_1^{[2]}$.
3) $Q_{k,i}^{[n+1]}=t^{n-i+3}P_{(n-2)\dots k}^{[n-1]}P_{n-i}^{[n-i+1]}-\sum_{j=i+1}^{n-1}t^{j-i}Q_{k,j}^{[n+1]}-\sum_{j=i}^{n-1}t^{j-i+2}Q_{k,j}^{[n]}P_{(n-i)\dots j-i+1}^{[n-i+1]}$, $i=1,\dots,n-2$.

Lemma 3 [11]. In $\mathbf{MB}_{n+1}$,

1) $Q_{n-1,n}^{[n+1]}=Q_{\phi,i}^{[n+1]}=0$.
2) $Q_{\phi,n}^{[n+1]}=t^3$.
3) $Q_{n-1,i}^{[n+1]}=Q_{n-i,1}^{[n-i+2]}, i=2,\dots,n-1$.
4) $Q_{*,n}^{[n+1]}=Q_{?,n}^{[n+1]}+Q_{n-2,n}^{[n+1]}$ and $Q_{*,i}^{[n+1]}=Q_{n-2,i}^{[n+1]}+Q_{n-1,i}^{[n+1]}$ for $i=1,\dots,n-1$.

The linear system for the series $P_*^{[n+1]}$ (corresponding to irreducible words) was also proved in [11] in the form of the following lemma.

Lemma 4 [11]. The following relations hold for the irreducible words in $\mathbf{MB}_{n+1}$.

1) $P_k^{[n+1]} = P_k^{[n]} P_n^{[n+1]} + P_k^{[n]}, \quad k = 1,\ldots,n-1.$

2) $P_n^{[n+1]} = tP_n^{[n+1]} + P_{n(n-1)}^{[n+1]} + t$.

3) $P_{(n-1)\cdots i}^{[n+1]} = P_{(n-1)\cdots i}^{[n]} P_n^{[n+1]} + P_{(n-1)\cdots i}^{[n]}, \quad i = 1,\ldots,n-2.$

4) $P_{n(n-1)}^{[n+1]} = tP_{n-1}^{[n+1]} - \sum_{j=1}^{n} t^{j-n-1} Q_{*,j}^{[n+1]} P_{n\cdots j}^{[n+1]}$.

5) $P_{n\cdots k}^{[n+1]} = tP_{(n-1)\cdots k}^{[n+1]} - \sum_{j=1}^{n} t^{j-n-1} Q_{k,j}^{[n+1]} P_{n\cdots j}^{[n+1]}$, $k = 1,\ldots,n-2$.

Using the above linear systems we had calculated (see details in [11]) the Hilbert series of $\mathbf{MB}_3$ and $\mathbf{MB}_4$ and their corresponding growth rate. The outline of the Hilbert series of $\mathbf{MB}_3$ is given in an example**Example 1 [11].** Note that the Hilbert series of the set $A_1^{[2]} = \{x_1, x_1^2, x_1^3, \ldots\}$ is given by $P_1^{[2]} = t + t^2 + t^3 + \cdots = \frac{t}{1-t}$. The only two types of reducible words in $\mathbf{MB}_3$ are $B_{?,2}^{[3]} = x_2 x_1 x_2$ and $B_{1,1}^{[3]} = \{x_2 x_1\} \times A_1^{[2]} \times \{x_2 x_1\}$. Therefore the corresponding Hilbert series are $Q_{\phi,2}^{[3]} = t^3$ and $Q_{1,1}^{[3]} = \frac{t^5}{1-t}$ respectively. Therefore

$$P_1^{[3]} = \frac{t}{1-t} + \frac{t}{1-t} P_2^{[3]}, \qquad P_2^{[3]} = t + P_{21}^{[3]} + tP_2^{[3]} \quad \text{and}$$

$$P_{21}^{[3]} = tP_1^{[3]} - t^2 P_2^{[3]} - \frac{t^3}{1-t} P_{21}^{[3]}.$$

Solving the above equations simultaneously we get

$$P_1^{[3]} = \frac{t}{(1-t)(1-t-t^2)}, \qquad P_2^{[3]} = \frac{t+t^2}{1-t-t^2},$$

$$P_{21}^{[3]} = \frac{t^2}{1-t-t^2}.$$

As we have $A^{[3]} = \{e\} \coprod A_1^{[3]} \coprod A_2^{[3]}$. Therefore the Hilbert series of $\mathbf{MB}_3$ is given by

$$\mathbf{H}_{\mathbf{M}}^{[3]}(t) = 1 + P_1^{[3]} + P_2^{[3]} = \frac{1}{(1-t)(1-t-t^2)}$$

$$= 1 + 2t + 4t^2 + 7t^3 + 12t^4 + 20t^5 + \cdots.$$

Remark 1. One can see that the coefficients $a_k^{[3]}$ in the above series are related with Fibonacci numbers $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, \ldots$ by the relation $a_k^{[3]} = F_{k+2} - 1$.

Remark 2. As we have $\frac{1}{(1-t)(1-t-t^2)} = \frac{-1}{1-t} + \frac{5-2\sqrt{5}}{5(1+c_1 t)} + \frac{5+2\sqrt{5}}{5(1-c_2 t)}$ where $c_1 = \frac{\sqrt{5}-1}{2}$ and $c_2 = \frac{\sqrt{5}+1}{2}$; the first two terms have a negligible contribution in approximating the series, while the last term $\frac{5+2\sqrt{5}}{5}\left(1 + c_2 t + c_2^2 t^2 + c_2^3 t^3 + \cdots\right)$ approximates the series. Hence $a_k^{[3]} \approx \frac{5+2\sqrt{5}}{5}\left(\frac{\sqrt{5}+1}{2}\right)^k$. Thus the growth function $a_k^{[3]}$ of $\mathbf{MB}_3$ is exponential and the growth rate is $\frac{\sqrt{5}+1}{2}$.

Similarly we had shown that

Example 2 [11]. The Hilbert series of $\mathbf{MB}_4$ is given by

$$\mathbf{H}_{\mathbf{M}}^{[4]}(t) = \frac{1}{(1-t)(1-2t-t^2+t^3+t^4+t^5)}$$

and the corresponding growth rate is 2.087.

The next result is a direct application of the Lemma 1, Lemma 2, Lemma 3 and Lemma 4.

Lemma 5 [12]. The Hilbert series of the braid monoid $\mathbf{MB}_5$ is given by $\mathbf{H}_{\mathbf{M}}^{[5]}(t) = \frac{1}{(1-t)(1-3t+3t^3+t^4+t^5-t^6-t^7-t^8-t^9)}$.

Corollary 1 [12]. The growth rate of $\mathbf{MB}_5$ is 2.395.

Lemma 6 [12]. The Hilbert series of the braid monoid $\mathbf{MB}_6$ is given by

$$\mathbf{H}_{\mathbf{MB}}^{[6]}(t) = \frac{1}{(1-t)(1-4t+2t^2+5t^3-t^4-t^5-3t^6-t^7-t^8-t^9+t^{10}+t^{11}+t^{12}+t^{13}+t^{14})}.$$

Corollary 2 [12]. The growth rate of $\mathbf{MB}_6$ is approximately equal to 2.6.

Now we have our main result.

Theorem 2. The Hilbert series $\mathbf{H}_{\mathbf{MB}}^{[7]}(t)$ of the braid monoid $\mathbf{MB}_7$ is given by

$$\frac{1}{(1-t)(1-5t+5t^2+6t^3-6t^4-3t^5-4t^6+2t^7+2t^8+3t^{10}+t^{11}+t^{12}+t^{13}+t^{14}-t^{15}-t^{16}-t^{17}-t^{18}-t^{19}-t^{20})}.$$

Proof. As above, using the results of the previous lemmas (Lemma 1, Lemma 2, Lemma 3) and of Theorem 5 and Theorem 6 we have the following coefficients of $P_*^{[7]}$ in simplified form:

$$Q_{*,6}^{[7]} = \frac{t^3}{T_9}(1 - 2t - t^2 + t^3 + t^4 + t^5).$$

$$Q_{*,5}^{[7]} = \frac{t^5}{T_5 T_9}(1 - 4t + 2t^2 + 6t^3 - t^4 - 3t^5 - 5t^6 - 2t^7 + 2t^8 + 3t^9 + 3t^{10} + t^{11}).$$

$$Q_{*,4}^{[7]} = \frac{t^7}{T_2^2 T_5 T_9}(1 - 5t + 4t^2 + 13t^3 - 14t^4 - 16t^5 + 8t^6 + 15t^7 + 10t^8 - 5t^9 - 11t^{10}$$

$-8t^{11}-4t^{12}+3t^{13}+5t^{14}+3t^{15}+t^{16})$

$Q^{[7]}_{*,3}=\frac{t^{9}}{T_1^2T_2^2T_5^2T_9}(1-8t+21t^{2}-8t^{3}-50t^{4}+55t^{5}+45t^{6}-53t^{7}-56t^{8}+4t^{9}+86t^{10}$
$+24t^{11}-56t^{12}-34t^{13}-2t^{14}+27t^{15}+19t^{16}+t^{17}-6t^{18}-14t^{19}-7t^{20}+2t^{21}$
$+5t^{22}+4t^{23}+t^{24})$.

$Q^{[7]}_{*,2}=\frac{t^{11}}{T_1^2.T_2.T_5^2.T_9^2}(1-9t+28t^{2}-21t^{3}-63t^{4}+115t^{5}+25t^{6}-117t^{7}-58t^{8}+41t^{9}+191t^{10}$
$-10t^{11}-145t^{12}-102t^{13}-19t^{14}+192t^{15}+82t^{16}-60t^{17}-78t^{18}-66t^{19}+33t^{20}$
$+29t^{21}+13t^{22}+11t^{23}-10t^{24}+3t^{26}+4t^{27}+2t^{28}-5t^{29}-5t^{30}-3t^{31}-t^{32})$.

$Q^{[7]}_{*,1}=\frac{t^{13}}{T_1^2.T_2.T_5.T_9^2.T_{14}}(1-10t+36t^{2}-40t^{3}-70t^{4}+200t^{5}-34t^{6}-244t^{7}+46t^{8}$
$+164t^{9}+208t^{10}-174t^{11}-278t^{12}-2t^{13}+46t^{14}+330t^{15}+151t^{16}$
$-223t^{17}-25t^{18}-123t^{19}+170t^{20}+148t^{21}+25t^{22}+10t^{23}-42t^{24}$
$-40t^{25}-18t^{26}-4t^{27}-9t^{28}-3t^{29}+16t^{30}+22t^{31}+10t^{32}-8t^{33}$
$-10t^{34}-8t^{35}-3t^{36}+3t^{37}+4t^{38}+3t^{39}+t^{40})$.

$Q^{[7]}_{1,6}=\frac{t^{7}}{T_9}(1-2t-t^{2}+t^{3}+t^{4}+t^{5})$.

$Q^{[7]}_{1,5}=\frac{t^{13}}{T_5T_9}(1-2t^{2}-t^{3}+t^{4}+2t^{5}+t^{6})$.

$Q^{[7]}_{1,4}=\frac{t^{16}}{T_2^2T_5T_9}(1-2t^{2}-2t^{5}-t^{6}+t^{7}+2t^{8}+t^{9})$.

$Q^{[7]}_{1,3}=\frac{t^{18}}{T_1^2T_2^2T_5^2T_9}(1-4t+4t^{2}+6t^{3}-12t^{4}-6t^{5}+12t^{6}+9t^{7}-2t^{8}-5t^{9}-3t^{10}$
$-4t^{11}-4t^{12}+t^{13}+4t^{14}+3t^{15}+t^{16})$.

$Q^{[7]}_{1,2}=\frac{t^{20}}{T_1^2.T_2.T_5^2.T_9^2}(1-5t+6t^{2}+12t^{3}-32t^{4}-2t^{5}+50t^{6}-8t^{7}-41t^{8}-10t^{9}+35t^{10}$
$+32t^{11}-27t^{12}-34t^{13}-7t^{14}+9t^{15}+18t^{16}+15t^{17}+9t^{18}-t^{19}-9t^{20}-8t^{21}-4t^{22}-t^{23})$.

$Q^{[7]}_{1,1}=\frac{t^{22}}{T_1^2.T_2.T_5.T_9^2.T_{14}}(1-6t+10t^{2}+10t^{3}-43t^{4}+5t^{5}+74t^{6}-114t^{8}-36t^{9}+151t^{10}$
$+66t^{11}-102t^{12}-94t^{13}-10t^{14}+73t^{15}+58t^{16}+21t^{17}-15t^{18}-57t^{19}$
$-27t^{20}+9t^{21}+22t^{22}+15t^{23}-4t^{24}-7t^{25}-7t^{26}-3t^{27}+3t^{28}+4t^{29}+3t^{30}+t^{31})$.

$Q^{[7]}_{2,6}=\frac{t^{6}}{T_9}(1-t-2t^{2}+t^{4}+t^{5}+t^{6})$.

$Q^{[7]}_{2,5}=\frac{t^{11}}{T_5T_9}(1-t^{2}-t^{3}-2t^{4}+2t^{6}+2t^{7}+t^{8})$.

$Q^{[7]}_{2,4}=\frac{t^{14}}{T_1T_2^2T_5T_9}(1-t-3t^{3}-t^{4}+5t^{5}+t^{6}+t^{8}-t^{10}-2t^{11}-t^{12})$.

$Q^{[7]}_{2,3}=\frac{t^{16}}{T_1^2T_2^2T_5^2T_9}(1+4t-5t^{2}-3t^{3}+13t^{4}-4t^{5}-4t^{6}-7t^{7}-8t^{8}+14t^{9}+9t^{10}$
$-2t^{12}-4t^{13}-t^{14}-t^{15}-t^{16})$.

$Q^{[7]}_{2,2}=\frac{t^{18}}{T_1^2.T_2^2.T_5^2.T_9^2}(1-5t+7t^{2}+8t^{3}-33t^{4}+25t^{5}+12t^{6}-31t^{7}+38t^{8}-12t^{9}$
$-26t^{10}+t^{11}+30t^{13}-6t^{14}-14t^{15}+2t^{16}-4t^{17}+6t^{18}+6t^{19}+5t^{20}$
$+2t^{21}-5t^{22}-5t^{23}-3t^{24}-t^{25})$.

$Q^{[7]}_{2,1}=\frac{t^{20}}{T_1^2.T_2^2.T_5.T_9^2.T_{14}}(1-6t+11t^{2}+5t^{3}-41t^{4}+35t^{5}+26t^{6}-52t^{7}+39t^{8}-$
$13t^{9}-63t^{10}+45t^{11}+70t^{12}+7t^{13}-89t^{14}-64t^{15}+50t^{16}+74t^{17}+24t^{18}$
$-14t^{19}-35t^{20}-42t^{21}-9t^{22}+21t^{23}+27t^{24}+12t^{25}-8t^{26}-10t^{27}-8t^{28}$
$-3t^{29}+3t^{30}+4t^{31}+3t^{32}+t^{33})$.

$Q^{[7]}_{3,6}=\frac{t^{5}}{T_9}(1-t-t^{2}-t^{3}+t^{5}+t^{6}+t^{7})$.

$Q_{3,5}^{[7]} = \frac{t^9}{T_1T_5T_9}(1-2t+t^4+2t^5-t^6+t^7-t^9-t^{10}-t^{11})$.

$Q_{3,4}^{[7]} = \frac{t^{12}}{T_1T_2^2T_5T_9}(1-t-4t^2+2t^3+6t^4+2t^5-2t^6-5t^7-5t^8-2t^9+2t^{10}+4t^{11}+3t^{12}+t^{13})$.

$Q_{3,3}^{[7]} = \frac{t^{14}}{T_1^2T_2^2T_5^2T_9}(1-5t+10t^2-7t^3-14t^4+32t^5+t^6-33t^7-11t^8+16t^9$
$+29t^{10}+3t^{11}-12t^{12}-9t^{13}-10t^{14}-3t^{15}+3t^{16}+5t^{17}+4t^{18}+t^{19})$.

$Q_{3,2}^{[7]} = \frac{t^{16}}{T_1^2T_2T_5^2T_9^2}(1-6t+13t^2-7t^3-17t^4+23t^5+t^6+20t^7-52t^8-49t^9$
$+105t^{10}+31t^{11}-38t^{12}-40t^{13}-30t^{14}+32t^{15}+8t^{16}+2t^{17}+10t^{18}$
$-t^{19}+2t^{20}+3t^{21}+4t^{22}+2t^{23}-5t^{24}-5t^{25}-3t^{26}-t^{27})$.

$Q_{3,1}^{[7]} = \frac{t^{18}}{T_1^2T_2T_5T_9^2T_{14}}(1-7t+18t^2-15t^3-16t^4+34t^5-9t^6+12t^7-42t^8$
$-23t^9+81t^{10}-3t^{11}-11t^{12}-31t^{13}-24t^{14}+43t^{15}-27t^{16}-18t^{17}+41t^{18}+$
$26t^{19}+7t^{20}-22t^{21}-27t^{22}-29t^{23}-6t^{24}+20t^{25}+25t^{26}+11t^{27}-8t^{28}-$
$10t^{29}-8t^{30}-3t^{31}+3t^{32}+4t^{33}+3t^{34}+t^{35})$.

$Q_{4,6}^{[7]} = \frac{t^4}{T_9}(1-t-2t^2+t^5+t^6+t^7+t^8)$.

$Q_{4,5}^{[7]} = \frac{t^7}{T_1T_5T_9}(1-3t+4t^3+2t^4-t^5-3t^6-2t^7-2t^8+2t^{10}+2t^{11}+t^{12})$.

$Q_{4,4}^{[7]} = \frac{t^{10}}{T_1T_2^2T_5T_9}(2-8t+4t^2+15t^3-7t^4-12t^5-7t^6+2t^7+10t^8+7t^9+4t^{10}-2t^{11}-5t^{12}-3t^{13}-t^{14})$.

$Q_{4,3}^{[7]} = \frac{t^{12}}{T_1^2T_2^2T_5^2T_9}(1-4t-t^2+19t^3-4t^4-42t^5+2t^6+59t^7+15t^8-44t^9$
$-35t^{10}+6t^{11}+31t^{12}+15t^{13}-2t^{14}-8t^{15}-13t^{16}-6t^{17}+2t^{18}+5t^{19}+4t^{20}+t^{21})$.

$Q_{4,2}^{[7]} = \frac{t^{14}}{T_1^2T_2T_5^2T_9^2}(1-6t+11t^2-4t^3+2t^4-12t^5-34t^6+70t^7+33t^8-34t^9$
$-94t^{10}-51t^{11}+153t^{12}+69t^{13}-52t^{14}-66t^{15}-48t^{16}+35t^{17}+20t^{18}+7t^{19}$
$+10t^{20}-8t^{21}+t^{22}+3t^{23}+4t^{24}+2t^{25}-5t^{26}-5t^{27}-3t^{28}-t^{29})$.

$Q_{4,1}^{[7]} = \frac{t^{16}}{T_1^2T_2T_5T_9^2T_{14}}(1-7t+15t^2+t^3-41t^4+35t^5+12t^6-11t^7+20t^8-36t^9$
$-26t^{10}-40t^{11}+94t^{12}+130t^{13}-83t^{14}-133t^{15}-43t^{16}+77t^{17}+54t^{18}-8t^{19}+$
$20t^{20}+9t^{21}-16t^{22}-15t^{23}-14t^{24}-22t^{25}-6t^{26}+17t^{27}+24t^{28}+11t^{29}-8t^{30}$
$-10t^{31}-8t^{32}-3t^{33}+3t^{34}+4t^{35}+3t^{36}+t^{37})$.

$Q_{5,5}^{[7]} = \frac{t^5}{T_1}$.

$Q_{5,4}^{[7]} = \frac{t^7}{T_1T_2}$.

$Q_{5,3}^{[7]} = \frac{t^9}{T_2T_5}(1-t^2)$.

$Q_{5,2}^{[7]} = \frac{t^{11}}{T_5T_9}(1-t-2t^2+t^3+2t^4+t^5-t^6-t^7)$.

$Q_{5,1}^{[7]} = \frac{t^{13}}{T_9T_{14}}(1-2t-2t^2+4t^3+3t^4-5t^6-4t^7+t^8+2t^9+4t^{10}+t^{11}-t^{12}-t^{13}-t^{14})$.

Now we have the system for irreducible words (of Lemma 4) of 12 equations and the same number of variables $P_*^{[7]}$:

$P_1^{[7]} = t(\frac{1+P_6^{[7]}}{T_1T_{14}})$.

$P_2^{[7]} = t(t+1)(\frac{1+P_6^{[7]}}{T_{14}})$.

$P_3^{[7]} = t(1-t^2-t^3-t^4)(\frac{1+P_6^{[7]}}{T_{14}})$.

$P_4^{[7]} = t(t^8+t^7+t^6+t^5-2t^2-t+1)(\frac{1+P_6^{[7]}}{T_{14}})$.

$P_5^{[7]} = t(1-2t-2t^2+2t^3+2t^4+2t^5-t^9-t^{10}-t^{11}-t^{12}-t^{13})(\frac{1+P_6^{[7]}}{T_{14}})$.

$P_6^{[7]} = tP_6^{[7]} + P_{65}^{[7]} + t$.

$P_{54321}^{[7]} = P_{54321}^{[6]}(1+P_6^{[7]})$.

$P_{54321}^{[7]} = t^5(1-3t+3t^3+t^4+t^5-t^6-t^7-t^8-t^9)(\frac{1+P_6^{[7]}}{T_{14}})$.

$P_{5432}^{[7]} = t^4(1-2t-2t^2+2t^3+3t^4+t^5-t^7-t^8-t^9-t^{10})(\frac{1+P_6^{[7]}}{T_{14}})$.

$P_{543}^{[7]} = t^3(1-2t-t^2+t^3+t^4+2t^5+t^6-t^8-t^9-t^{10}-t^{11})(\frac{1+P_6^{[7]}}{T_{14}})$.

$P_{54}^{[7]} = t^2(1-2t-t^2+t^3+t^4+t^5+t^6+t^7-t^9-t^{10}-t^{11}-t^{12})(\frac{1+P_6^{[7]}}{T_{14}})$.

$P_{654321}^{[7]} = tP_{54321}^{[7]} - t^{-6}Q_{1,1}^{[7]}P_{654321}^{[7]} - t^{-5}Q_{1,2}^{[7]}P_{65432}^{[7]} - t^{-4}Q_{1,3}^{[7]}P_{6543}^{[7]} - t^{-3}Q_{1,4}^{[7]}P_{654}^{[7]} - t^{-2}Q_{1,5}^{[7]}P_{65}^{[7]} - t^{-1}Q_{1,6}^{[7]}P_6^{[7]}$.

$P_{65432}^{[7]} = tP_{5432}^{[7]} - t^{-6}Q_{2,1}^{[7]}P_{654321}^{[7]} - t^{-5}Q_{2,2}^{[7]}P_{65432}^{[7]} - t^{-4}Q_{2,3}^{[7]}P_{6543}^{[7]} - t^{-3}Q_{2,4}^{[7]}P_{654}^{[7]} - t^{-2}Q_{2,5}^{[7]}P_{65}^{[7]} - t^{-1}Q_{2,6}^{[7]}P_6^{[7]}$

$P_{6543}^{[7]} = tP_{543}^{[7]} - t^{-6}Q_{3,1}^{[7]}P_{654321}^{[7]} - t^{-5}Q_{3,2}^{[7]}P_{65432}^{[7]} - t^{-4}Q_{3,3}^{[7]}P_{6543}^{[7]} - t^{-3}Q_{3,4}^{[7]}P_{654}^{[7]} - t^{-2}Q_{3,5}^{[7]}P_{65}^{[7]} - t^{-1}Q_{3,6}^{[7]}P_6^{[7]}$.

$P_{654}^{[7]} = tP_{54}^{[7]} - t^{-6}Q_{4,1}^{[7]}P_{654321}^{[7]} - t^{-5}Q_{4,2}^{[7]}P_{65432}^{[7]} - t^{-4}Q_{4,3}^{[7]}P_{6543}^{[7]} - t^{-3}Q_{4,4}^{[7]}P_{654}^{[7]} - t^{-2}Q_{4,5}^{[7]}P_{65}^{[7]} - t^{-1}Q_{4,6}^{[7]}P_6^{[7]}$.

$P_{65}^{[7]} = tP_5^{[7]} - t^{-6}Q_{*,1}^{[7]}P_{654321}^{[7]} - t^{-5}Q_{*,2}^{[7]}P_{65432}^{[7]} - t^{-4}Q_{*,3}^{[7]}P_{6543}^{[7]} - t^{-3}Q_{*,4}^{[7]}P_{654}^{[7]} - t^{-2}Q_{*,5}^{[7]}P_{65}^{[7]} - t^{-1}Q_{*,6}^{[7]}P_6^{[7]}$.

Now solving this system for the variables the above equations simultaneously for the variables, $P_j^{[7]}; j = 54321, 5432, 543, 54, 654321, 65432, 6543, 654, 65, 6, 5, 4, 3, 2, 1$ we get the following results: let $T_{20} = 1-5t+5t^2+6t^3-6t^4-3t^5-4t^6+2t^7+2t^8+3t^{10}+t^{11}+t^{12}+t^{13}+t^{14}-t^{15}-t^{16}-t^{17}-t^{18}-t^{19}-t^{20}$. Then we have

$P_1^{[7]} = \frac{t}{T_1.T_{20}}$, $P_2^{[7]} = \frac{t(t+1)}{T_{20}}$, $P_3^{[7]} = \frac{(t-t^3-t^4-t^5)}{T_{20}}$, $P_4^{[7]} = \frac{t(t^8+t^7+t^6+t^5-2t^2-t+1)}{T_{20}}$.

$P_5^{[7]} = \frac{t(1-2t-2t^2+2t^3+2t^4+2t^5-t^9-t^{10}-t^{11}-t^{12}-t^{13})}{T_{20}}$.

$P_6^{[7]} = \frac{t(1-3t-t^2+5t^3+2t^4+t^5-3t^6-3t^7-t^8-2t^9+t^{14}+t^{15}+t^{16}+t^{17}+t^{18}+t^{19})}{T_{20}}$.

$P_{54321}^{[7]} = \frac{t^5(1-3t+3t^3+t^4+t^5-t^6-t^7-t^8-t^9)}{T_{20}}$.

$P_{5432}^{[7]} = \frac{t^4(1-2t-2t^2+2t^3+3t^4+t^5-t^7-t^8-t^9-t^{10})}{T_{20}}$.

$P_{543}^{[7]} = \frac{t^3(1-2t-t^2+t^3+t^4+2t^5+t^6-t^8-t^9-t^{10}-t^{11})}{T_{20}}$.

$P_{54}^{[7]} = \frac{t^2(1-2t-t^2+t^3+t^4+t^5+t^6+t^7-t^9-t^{10}-t^{11}-t^{12})}{T_{20}}$.

$P_{654321}^{[7]} = \frac{t^6(1-4t+2t^2+5t^3-t^4-t^5-3t^6-t^7-t^8-t^9+t^{10}+t^{11}+t^{12}+t^{13}+t^{14})}{T_{20}}$. $P_{65432}^{[7]} = \frac{t^5(1-3t-t^2+5t^3+3t^4-t^5-3t^6-3t^7-t^8-t^9+t^{11}+t^{13}+t^{14}+t^{15})}{T_{20}}$.

$P_{6543}^{[7]} = \frac{t^4(1-3t+3t^3+2t^4+t^5-2t^6-2t^7-2t^8-t^9-t^{10}+t^{12}+t^{13}+t^{14}+t^{15}+t^{16})}{T_{20}}$.

$P_{654}^{[7]} = \frac{t^3(1-3t+4t^3-t^7-t^8-2t^9-t^{10}-t^{11}+t^{13}+t^{14}+t^{15}+t^{16}+t^{17})}{T_{20}}$.

$P_{65}^{[7]} = \frac{t^2(1-3t+3t^3+2t^4-2t^6-t^8-t^9-t^{10}-t^{11}-t^{12}+t^{14}+t^{15}+t^{16}+t^{17}+t^{18})}{T_{20}}$.

Now we have $A^{[7]} = \{e\} \coprod A_1^{[7]} \coprod A_2^{[7]} \coprod A_3^{[7]} \coprod A_4^{[7]} \coprod A_5^{[7]} \coprod A_6^{[7]}$. Therefore

$\mathbf{H}_{\mathbf{MB}}^{[7]}(t) = 1 + \sum P_i^{[7]}; 1 \le i \le 6.$

Substituting all the required values of the irreducible words we get the following Hilbert series of braid monoid $\mathbf{MB}_7$ as:

$$\mathbf{H}_{\mathbf{MB}}^{[7]}(t) = \frac{1}{(1-t)\left(1-5t+5t^2+6t^3-6t^4-3t^5-4t^6+2t^7+2t^8+3t^{10}+\sum_{i=11}^{14} t^i - \sum_{i=15}^{20} t^i\right)}.$$

Corollary 3. The growth rate of $\mathbf{MB}_7$ is approximately equal to 2.74.

Proof. From the approximated partial fraction (again using Mapple) of

$$\frac{1}{(1-t)(1-5t+5t^2+6t^3-6t^4-3t^5-4t^6+2t^7+2t^8+3t^{10}+t^{11}+t^{12}+t^{13}+t^{14}-t^{15}-t^{16}-t^{17}-t^{18}-t^{19}-t^{20})}$$

we see that the term $\frac{8.8136}{1-2.7397t}$ from the above Hilbert Series has considerable coefficients in its expansion

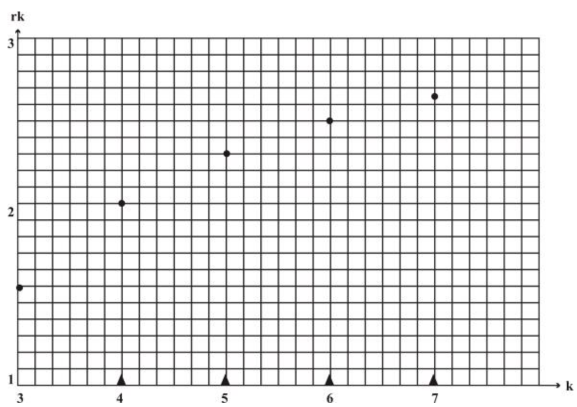
$$8.8136\left(1+2.7397t+(2.7397)^2t^2+(2.7397)^3t^3+\cdots\right).$$

Therefore the growth function $a_k^{[7]} \approx 8.8136(2.7397)^k$ and hence the growth rate of $\mathbf{MB}_7$ is approximately equal to 2.74.

At the end we are giving two conjectures about the degree of a polynomial involved in the Hilbert series of $\mathbf{MB}_{n+1}$ and about the growth rate of braid monoid $\mathbf{MB}_{n+1}$.

Conjecture 1: The degree of the polynomial (other than $1-t$) in the denominator of the Hilbert series of $\mathbf{MB}_{n+1}$ is $\frac{n(n+1)}{2}-1$.

Let r_k be the growth rate of $\mathbf{MB}_k$, then we have the following growth rates: $r_3 = 1.6$, $r_4 = 2.08$, $r_5 = 2.39$, $r_6 = 2.6$ and $r_7 = 2.74$. The following graph shows the values of r_k.



Conjecture 2: The upper bound for the growth rate of the braid monoid $\mathbf{MB}_{n+1}$ is 3.

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