A MODIFIED LINEARIZED IMPLICIT ITERATION METHODS FOR NON-SYMMETRIC ALGEBRAIC RICCATI EQUATIONS

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ABSTRACT: *Recently, a new linearized implicit iteration method (LI) has been proposed for the minimal non-negative solution of non-symmetric algebraic Riccati equation. In this research, we have introduced a modified form of the Linearized implicit iteration method (MLI) to solve the non-symmetric algebraic Riccati equation by using another parameter in a linear matrix equation, and built convergence analysis under suitable conditions. Numerical experiments proved that our modification is much more feasible and effective in the contrast to the exiting LI iteration method.*

Key words: Non-symmetric algebraic Riccati equations, LI method, M-matrix.

INTRODUCTION

Considering the problem of numerical solution to the non-symmetric algebraic Riccati equation (NARE)

$$XCX - XD - AX - B = 0, \qquad (1),$$

where $A \in \mathbb{R}^{mxm}$, $B \in \mathbb{R}^{mxn}$, $C \in \mathbb{R}^{nxm}$, and $D \in \mathbb{R}^{nxn}$

Such NARE has been introduced in transport theory, applied probability, Wiener-Hopf factorization of Markov chains and etc. The minimal non-negative solution is a practical interest. For theoretical background we refer to [4–8].Some notations and Preliminaries are given in the following.

At the first instance, we analysis some basic results of the M-matrices. For any matrices $A, B \in \mathbb{R}^{nxn}$, we write $A \geq B (A > B)$, if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$, for all i, j. A is called a Z-matrix if $a_{ij} \leq 0$, for all $i = j$. A Z-matrix A is called an M- matrix if there exists a non-negative matrix B with spectral radius $\rho(B)$ such as $A = sI - B$and $s \geq \rho(B)$. In particular A is said to be a non-singular M-matrix if $\rho(B) < s$ and singular M-matrix if $\rho(B) = s$.

Lemma 1.1. The following statements[1,3] are equivalent, if A be a Z-matrix

(1) A is a non-singular M-matrix;

(2) $A^{-1} \geq 0$;

(3) $Av > 0$ for some vectors $v > 0$;

(4) All eigenvalues of A have positive real part.

Lemma 1.2. [9, 10] Let A, B be Z-matrices. If A is a non-singular M-matrix and $A \leq B$ then B is also a non-singular M-matrix. In specific, for any non-negative real numberα, $B = \alpha I + A$ is a non-singular M-matrix.

Lemma 1.3. [10] Let N be a non-singular M-matrix or an irreducible singular M-matrix.Partition of N as

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

where N_{11} and N_{22} are square matrices. then $N_{11}$and N_{22} are non-singular M-matrices.

For existence of the minimal non-negative solution of the NARE associated with M- matrix, we have the following basic result [5, 6,10,11]

Lemma1.4. Let

$$K = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \qquad (2)$$

$A - SC$ and $D - CS$ are also a non-singular M-matrices, if eq (1) has a minimal non-negative solution S and K is a non-singular M-matrix.

Many methods have been proposed to attain the minimal non-negative solution of NARE, for example fixed-point iteration, Newton iteration, doubling algorithm,Alternating-Directional doubling algorithm, matrix sign function, alternately linearized implicit iteration and so on. Further details can be found in [5, 9, 11- 18].

For minimal non-negative solutions of non-symmetric algebraic Riccati equation (1), recently a new LI iteration method is proposed in [2].

The method is as follows:

- Set $X_0 = 0 \in \mathbb{R}^{mnx}$.
- compute X_{k+1} from X_k

$$\left(\alpha I + \left(A - X_k C\right)\right) X_{k+1} = X_k \left(\alpha I - D\right) + B,$$

where $k = 0,1, \ldots, until$ {Xk } converge.

We have another new LI iteration method: from $X_k$by solving $X_{k+1}$from the following

$$X_{k+1}\left(\alpha I + \left(D - CX_k\right)\right) = \left(\alpha I - D\right) X_k + B,$$

where $\alpha > 0$ is a parameter

The following convergence theorem is obtained in [11].

Theorem 1.1. For K is a non-singular M-matrix in (2), S be a minimal non-negative solution of non-symmetric algebraic Riccati equation (1). The initial matrix $X_0 = 0$ and the parameter α satisfies

$$\alpha \geq \max\left\{ \max_{1 \leq i \leq m} \{a_{ii}\}, \max_{1 \leq j \leq n} \{d_{jj}\} \right\},$$

The matrix sequence $\{X_k\}$ matrix sequence produced by new LI iteration method is well defined, monotonically increasing and converges to S.

Modified new LI iteration method's main purpose is that, it has fast convergence rate, comparable computing cost in contrast to Newton method and all fixed-point iteration methods. In each iteration step, itneedsonly to solve a linear matrix equation. Hence, there could be a vast difference between the magnitudes of the matrices A and D in some applications, where the LI method may not be effective anymore in these cases. As an example, by considering the following NARE

$$A = 180105.I_{18} - 10^4.1_{18,18}, B = 1_{18,2}, C = B^T, D = 18.I_2,$$

where I representing identity matrix and 1 is a matrix of all ones.

This example is from [16] where the original matrix A $= 180105I_{18} - 10^4.1_{18,18}$ .We change the entry of A such that the corresponding K is a non-singular M-matrix. For such a simple NARE, we find that the LI method cannot converge in 10000 iterations in our experiment. It means that the LI method fail in this example. So in the next section, we propose to modify LI method by adding another parameter in the linear matrix equation, and develop convergence analysis. The rest of the paper is organized as follows. In section 2 we propose the modified LI method and its convergence analysis. In Section 4, numerical experiments confirm the usefulness of our modified method and finally summarizing remarks given in section 5.

MODIFIED NEW *LI* ITERATION METHOD

From the LI method and example as described above, there is a vast difference between the matrices A and D. It seems that there could be two different iterative direction of A and D. Thus the LI method can be modified as follows.

Algorithm

1: Set $\chi_0 = 0 \in \mathbb{R}^{mnx}$,

2: Compute χ_{k+1} from χ_k

Herein, two cases arises:

3: Case (1) If magnitude of matrix D is very large as compared matrix A, Then the Iteration process takes the form

$$\chi_{k+1}\left(\alpha I + \left(D - C\chi_k\right)\right) = \left(\alpha I - D\right)\chi_k + B \qquad (3)$$

And/ Or

4: Case (2) If magnitude of matrix A is very large as compared matrix D, then the iteration process takes the form:

$$\left(\beta I + \left(A - \chi_k C\right)\right)\chi_{k+1} = \chi_k\left(\beta I - D\right) + B \qquad (4)$$

where $\alpha > 0, \beta > 0$ are two parameters.

CONVERGENCE ANALYSIS

Convergence analysis of the modified new LI method given as follows.

Lemma 3.1. Suppose the matrix sequences $\{X_k\}$ be generated by the modified new LI iteration method, and S be the minimal non-negative solution of (1)

i-e. $\mathfrak{R}(X) = XCX - XD - AX + B$.

Then the following equalities satisfied:

(i) $\left(\beta I + \left(A - X_k C\right)\right)\left(X_{k+1} - S\right) = \left(X_k - S\right)\left(\beta I - \left(D - CS\right)\right)$;

(ii) $\left(\beta I + \left(A - X_k C\right)\right)\left(X_{k+1} - X_k\right) = \mathrm{R}\left(X_k\right)$;

(iii) $\mathrm{R}\left(X_{k+1}\right) = \left(X_{k+1} - X_k\right)\left(\beta I - \left(D - CX_{k+1}\right)\right)$.

Proof: From the above lemma followed by [11]. Therefore, the proof has been omitted here.

Lemma 3.2. Let the matrix sequences $\{X_k\}$ be generated by the modified new LI, and S be the minimal non-negative solution of (1) **i-e.,**

$\mathfrak{R}(X) = XCX - XD - AX + B$,

Then the following equalities hold**:**

(i) $\left(X_{k+1} - S\right)\left(\alpha I + \left(D - CX_k\right)\right) = \left(\alpha I - \left(A - SC\right)\right)\left(X_k - S\right)$;

(ii) $\left(X_{k+1} - X_k\right)\left(\alpha I + \left(D - CX_k\right)\right) = \mathrm{R}\left(X_k\right)$;

(iii) $\mathrm{R}\left(X_{k+1}\right) = \left(\alpha I - \left(A - X_{k+1}C\right)\right)\left(X_{k+1} - X_k\right)$;

Proof: The proof of above lemma is same as Lemma (3.1). Using above Lemmas, we can prove the following convergence theorem of the modified new LI iteration method.

Theorem 3.1. Let S be the minimal non-negative solution of non-symmetric algebraic Riccati equation (1) and K is a non-singular M-matrix in (2), and the parameters α, satisfying:

$$\alpha \ge \max_{1 \le i \le m}\{a_{ii}\},$$

where a_{ii} is the ith diagonal element of matrix A, the $\{X_k\}$ generated by modified new LI iteration is well defined and satisfied.

(a) The matrix sequence $\{X_k\}$ is monotonically increasing and bounded i.e.,

$$0 \le X_0 \le X_1 \le \ldots \le X_k \le X_{k+1} \le \ldots S$$

(b) $\lim_{k\to\infty} X_k = S$ i-e the matrix sequence $\{X_k\}$ is convergent to S.

Proof: If K is a non-singular M-matrix, B, C ≥ 0 and A and D are non-singular M-matricesfrom lemma 1.3.

The matrix $\alpha I - A$ is non-negative matrix, when $\alpha \ge max_{1\le i\le m}\{a_{ii}\}$.For the $\{X_k\}$ generated by modified new LI iteration, we first show that the following facts hold for $k = 0,1, \ldots$

The result of (a) is equivalent to the following conclusion:

(i) 0, $\le X_0 \le X_k \le X_{k+1} \le S$, $\mathcal{R}(X_k) \ge 0$, and $\mathcal{R}(X_{k+1}) >$, 0, k = 0, 1, 2…,

(ii) $D - CX_{k+1}$is non-singular M-matrix.

By using induction we prove the above results. Since$X_0 = 0$, when$k = 0$, we have $\mathcal{R}(X_0) = B \ge 0$, from modified new LI iteration. Process (3), we have $X_1\left(\alpha I + D\right) = B$. As D is a non-singular M-matrixfrom Lemma 1.2, $\alpha I + D$ is also nonsingular M-matrix.

Thus from Lemma 1.1 we have,$(\alpha I + D)^{-1} \ge 0$.

Hence,

$$X_1 = B\left(\alpha I + D\right)^{-1} \ge 0 = X_0.$$

On the other hand, from lemma 2.2 (i), we have

$$\left(X_1 - S\right)(\alpha I + D) = (\alpha - (A - SC))(-S)$$

Thus

$$\left(X_1-S\right)=-(\alpha I-(A-SC))S(\alpha I+D)^{-1}$$

$$\left(X_1-S\right)=-((\alpha I-A)+SC)S(\alpha I+D)^{-1}\leq 0$$

From Lemma 2.2(iii)

$$\mathcal{R}(X_1)=\left(aI-(A-X_1C)\right)X_1$$

= ((aI-A) +X1C) X1 ≥ 0

This shows that

$$X_0\leq X_1\leq S, \mathrm{R}\,(X_0)\geq 0, \mathrm{R}\,(X_1)\geq 0.$$

We have $D-CS\leq D-CX_1\leq D,$ thus, by Lemma 1.2 $D-CX_1$ is a non- singular M-matrix.

Suppose that conclusion is true for $k = l-1$.

i.e., $0\leq X_0\leq X_{l-1}\leq X_l\leq S,\quad \mathrm{R}\,(X_{l-1})\geq 0$ and $\mathrm{R}\,(X_l)\geq 0$.

Since C ≥ 0, it follows that $D-CS\leq D-CX_l\leq D$

By using Lemma 1.2, since D - CX_1 is non-singular M-matrix.

From the iteration process (3), we have

$$X_{l+1}(\alpha I+(D-CX_l))=(\alpha I-A)X_l+B,$$

Thus

$$X_{l+1}=((\alpha I-A)X_l+B)(\alpha I+(D-CX_l))^{-1}\geq 0,$$

From Lemma 2.2(i), we have

$(X_{l+1}-S)=(\alpha I-(A-SC)(X_l-S)(\alpha I+(D-CX_l))^{-1}$ We know that $(\alpha I+(D-CX_l))^{-1}\geq 0$

$$=(\alpha I-A)+SC)(X_l-S)(\alpha I+(D-CX_l))^{-1}\leq 0$$

From Lemma 2.2(ii)

$$X_{l+1}-X_l=\mathrm{R}\,(X_l)(\alpha I+(D-CX_l))^{-1}\geq 0$$

From Lemma 2.2(iii)

$$\mathrm{R}\,(X_{l+1})=(\alpha I-(A-X_{l+1}C))(X_{l+1}-X_l)$$

$$=((\alpha I-A)+X_{l+1}C)(X_{l+1}-X_l)\geq 0$$

This confirms that

$0\leq X_l\leq X_{l+1}\leq S$, $R(X_l)\geq 0$, and $R(X_{l+1})\geq 0$.

From $C\geq 0$, then it follows,we have D − CS ≤ D − $CX_{l+1}\leq$ D.Thus by Lemma 1.2, D − $CX_{l+1}$is nonsingular M-matrix.Thus we have proved by induction that assertion (i) and (ii) holds true for k ≥ 0.

Under the above analysis we come to prove (b)

We have sequence of matrix $\{X_k\}$ is non-negative, bounded form above and monotonically increasing.

Thus there exist a non-negative matrix S∗, so lim k→∞ X k = S∗. From assertion (a)S∗ ≤ S. On the iteration process take limit on the other hand, we have solution of NARE is S∗,thus S ≤ S∗, because of minimal property of S, Hence S = S∗.

Theorem 3.2. Let S be the minimal non-negative solution of non-symmetric algebraic Riccati equation (1) and K is a non-singular M-matrix in (2), and the parameterβ satisfies

$$\beta\geq \max_{1\leq j\leq n}\{d_{jj}\},$$

where $d_1, d_2, \dots d_n$ are the diagonal element of the matrix D, at that time $\{X_k\}$ is sequence of matrix generated by modified new LI iteration is well-defined and it satisfy

(a) The matrix sequence $\{X_k\}$ is monotonically increasing and bounded.

$$0\leq X_0\leq X_1\leq\ldots\leq X_k\leq X_{k+1}\leq\ldots S$$

(b) lim k→∞ X_k = S i-e the matrix sequence $\{X_k\}$ is convergent to S.

Proof: From lemma 1.2 A and D are also non-singular M-matrice, if K is a non-singular M-matrixand B, C ≥ 0.

When $\beta\geq max_{1\leq j\leq n}\{d_{jj}\}$, and $\beta I-D$ is non-negative matrix.For the matrix sequences $\{X_k\}$ generated by the modified LI method, we first show that the following fact hold for k = 0,1,⋯

The result of (a) is alike to the following results:

(i) $0\leq X_0\leq X_k\leq X_{k+1}\leq S, \mathrm{R}\,(X_k)\geq 0,$ *and* $\mathrm{R}\,(X_{k+1})\geq 0$ $k=0,1,2\ldots$

(ii) $A-X_{1+1}C$ is nonsingular

By induction we prove the above results. Since$X_0=0$, when k=-0 we have R (X 0) = B ≥ 0 and from modified new LI iteration process (4).

$(\beta I+A)X_1=B$, since A is a non-singular M-matrix, from lemma 1.2 βI + A is also non-singular M-matrix.

Thus from Lemma 1.1 we have$(\beta I+A)^{-1}\geq 0$.

Hence,

$$X_1=(\beta I+A)^{-1}B\geq 0=X_0.$$

On the other hand, from lemma 2.1(i), we have

$$(\beta I+A)(X_1-S)=(-S)(\beta I+(D-CS))$$

Thus

$$X_1-S=-(\beta I+A)^{-1}S(\beta I+(D-CS))$$
$$=-(\beta I+A)^{-1}S((\beta I+D)-CS)$$
$$\leq 0$$

From Lemma 2.1(iii)

$$R(X_1)=X_1(\beta I-(D-CX_1))$$
$$=X_1(\beta I-D)+CX_1)$$
$$\geq 0$$

This shows that

$$X_0\leq X_1\leq S, \mathrm{R}\,(X_0)\geq 0, \textit{ and }\mathrm{R}\,(X_1)\geq 0.$$

We have $A-SC\leq A-X_1C\leq A,$

Thus, by Lemma 1.2 $A-X_1C$ is a non-singular M-matrix.

Suppose that result is true for $k=l-1$,

i.e., $0\leq X_0\leq X_l\leq X_{l+1}\leq S,\ \mathrm{R}\,(X_{l-1})\geq 0,$ and $R(X_l)\geq 0$.

Since C ≥ 0, so it follows that

$$A-SC\leq A-X_lC\leq A$$

Since D – CX_l is non-singular M-matrix by lemma 1.2.

From the iteration process (4), we have

$$(\beta I + (A - X_l C))X_{l+1} = X_l(\beta I - D) + B$$

Thus

$$X_{l+1} = (\beta I + (A - X_l C)^{-1} X_l(\beta I - D) + B$$
$$= (\beta I + A - X_l C)^{-1} X_l(\beta I - D) + B$$
$$\geq 0$$

From the Lemma 2.1 (i), we have

$$(\beta I + (A - X_l C))(X_{l+1} - S) = (X_l - S)(\beta I - (D - CS))$$

Thus

$$(X_{l+1} - S) = (\beta I + (A - X_l C))^{-1}(X_l - S)(\beta I - D) + CS \leq 0$$

From Lemma 2.1(ii), we have

$$(X_{l+1} - X_l) = (\beta I + (A - X_l C))^{-1} R(X_l)$$
$$\geq 0$$

From Lemma 2.1 (iii), we have

$$R(X_{l+1}) = (X_{l+1} - X_l)(\beta I - (D - CX_{l+1}))$$
$$\geq 0$$

This shows that

$$0 \leq X_1 \leq X_{l+1} \leq S, \mathrm{R}\,(X_l) \geq 0, \; and \; \mathrm{R}\,(X_{l+1}) \geq 0.$$

As we know C ≥ 0, we have

$$A - SC \leq A - X_{1+1}C \leq A.$$

$A - X_{l+1}C$is non-singular M-matrix by Lemma 1.2.

Thus we have proved by induction that assertion (i) and (ii) holds true for k ≥ 0.Under the above analysis we come to prove (b). We have $\{X_k\}$ sequence of matrix is non-negative, bounded form above. and monotonically increasing. Thus there exist a non-negative matrix S∗ such that lim k→∞ X_k= S∗. From statement (a) S∗ ≤ S. On the iteration process apply limit in the other hand, we have solution of NARE that is S*, so S ≤ S*, because of minimal property of S, Hence S = S*.

NUMERICAL EXPRIEMENTS

Here numerical effectiveness and performance of modified new LI method compared with LI method has been shown by test few examples. We present computational results in terms of the numbers of iterations, residue and CPU time. The residue is defined to be as in [1].

$$res = \frac{\|XCX - XD - AX + B\|_\infty}{\|XCX\|_\infty + \|XD\|_\infty + \|AX\|_\infty + \|B\|_\infty}$$

In our executions all iterations are run in MATLAB 2007b on a personal computer CORE i5 and are ended when the given iterate fulfills,

$$\frac{\|X_k C X_k - X_k D - AX_k + B\|_\infty}{\|B\|_\infty} < 1e - 6.$$

Experiment 1.Consider the NARE with

$$A = 180105.I_{18} - 10^4.1_{18\times18}$$
$$B = 1_{18,2}, C = B^T, D = 18.I_2,$$

where I is identity matrix and 1 is represent a matrix with all ones.Computational result summarized in the following table:

Table 1. Computational results of example 1

Method	Iteration number	CPU time	residue
LI	-	-	-
MLI	3	0.000449	6.7854e-009

In above experiment, from the computational results we can see that the modified LI is better than LI method.From the above numerical examples, we can see that the larger the difference between the matrices A and D, the better the modified LI comparing with LI method. If the magnitude of matrix D is larger than matrix A, then first iteration process (3) of modified LI can be used, otherwise, second iteration process (4) of modified LI method can work.

Experiment 2. Consider the NARE with

$$A = \begin{pmatrix} 0.5 & -0.1 \\ -0.1 & 0.5 \end{pmatrix}, \; B = \begin{pmatrix} 0.15 & 0.15 \\ 0.29 & 0.1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.19 & 0.10 \\ 0.19 & 0.10 \end{pmatrix}, D = \begin{pmatrix} 300 & -298 \\ -298 & 300 \end{pmatrix},$$

We have the computational results summarized in the following Table:

Table 2. Computational results of example 2

Method	Iteration number	CPU time	residue
LI	1770	0.025083	4.8782e-007
MLI	5	0.000459	2.6909e-007

From the computational results in the above experiment, we can see that the modified LI is better than LI method.

From the above numerical examples, we can see that the larger the difference between the matrices A and D, the better the modified LI comparing with LI method. If the magnitude of matrix D is larger than matrix A, then first iteration process (3) of modified LI can be used, otherwise, second iteration process (4) of modified LI method can work.

Experiment 3. Consider NARE with(n = 200)

$$A = \begin{pmatrix} 3 & -1 & & & \\ & 3 & -1 & & \\ & & \ddots & \ddots & \\ & & & 3 & -1 \\ & & & & 3 \end{pmatrix}$$

$$D = \xi A, B = 0.5 I_n, C = I_n,$$

where ξ is a positive constant. The computational results summarized in the following.

Table 3. Computational results of example 3

	Method	Iteration number	CPU time	residue
ξ = 100	LI	13	0.5236	2.8789e-07
	MLI	03	0.0666	5.980e-08
ξ = 500	LI	13	0.50528	3.08e-07
	MLI	02	0.05581	4.9801e-07
ξ = 1000	LI	13	0.48754	3.1097e-07
	MLI	2	0.052709	1.247e-07

Here, we have different values of ξ = 100, ξ = 500, ξ = 1000 from the computational results we can see that the modified LI is better than LI method.

From the above numerical experiment, we can see that the larger the difference between the matrices A and D, the better the modified LI comparing with LI method. If the magnitude of matrix D is larger than matrix A, then first iteration process (3) of modified LI can be used, otherwise, second iteration process (4) of modified LI method can work.

Example 4. Consider NARE with

$$A=\begin{pmatrix} 3 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 2 \end{pmatrix},$$

$$B=\begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & \\ & & & 1 \\ & & & 1 \end{pmatrix},$$

$$C=\begin{pmatrix} -1 & & & \\ -1 & -1 & & \\ & \ddots & \ddots & \\ & & -1 & -1 \\ & & & \end{pmatrix},$$

$$D=\begin{pmatrix} n+1 & -1 & \cdots & -1 \\ -1 & n+1 & \ddots & -1 \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n+1 \end{pmatrix},$$

:

Table 4. Computational results of example 4

	Method	Iteration number	CPU time	residue
n= 32	LI	109	0.01799	2.9530e-07
	MLI	15	0.002370	2.8489 e-07
n= 64	LI	219	0.149269	3.0496 e-07
	MLI	16	0.011444	1.5762 e-07
n= 128	LI	440	2.00058	3.0319 e-07
	MLI	16	0.06991	1.8845 e-07
n= 256	LI	882	3.2497	3.0318 e-07
	MLI	16	0.47671	2.0634 e-07
n= 512	LI	1767	372.74	3.0103 e-07
	MLI	16	3.643299	2.1598 e-07

where A,B,C,D are all of size n × n. This example is from [4, Chapter 3.6] where we change the (1,1) entry of Dfrom n to n+1 such that the corresponding K is a non-singular M-matrix. We have the computational results summarized in the following

In above experiment at different values of n, from the computational results we can see that the modified new LI is better than LI method.

From the above numerical example, we can see that the larger the difference between the matrices A and D, the better the modified LI comparing with LI method. If the magnitude of matrix D is larger than matrix A, then first iteration process (3) of modified LI can be used, otherwise, second iteration process (4) of modified LI method can work.

CONCLUSIONS

We have proposed a modified new LI method for non-symmetric algebraic Riccati equation associated with non-singular M-matrix by using one linear matrix equation and adding one parameter in the LI method. The convergence of modified new LI method is guaranteed as the LI method. Numerical experiments have shown that our method is effective and the improvementof the LI method.

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