ON JORDAN * CENTRALIZERS IN SEMIPRIME Γ-RINGS WITH INVOLUTION

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ABSTRACT: *In this paper we prove that if M is a 2-torsion free semiprime Γ-ring with involution satisfying $x\gamma y\alpha z = x\alpha y\gamma z$ and $f: M \to M$ an additive mapping such that $2f(x\beta x) = f(x)\beta x^* + x^*\beta f(x)$ for all $x \in M$, $\beta \in \Gamma$, then f is a Jordan * -centralizer.*

Keywords: Semiprime Γ-ring, involution, * -derivation, Jordan * -derivation, left (right) Jordan * -centralizer, Jordan * -centralizer, commutators.
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1 INTRODUCTION

Nobusawa [1] introduced the notion of a Γ-ring, a notion more general than a ring. Barnes [2] slightly weakened the conditions in the definition of a Γ-ring given by Nobusawa. After the study of Γ-rings by Nobusawa [1] and Barnes [2] many researchers have done a lot of work on Γ-rings and have obtained some generalizations of the corresponding results in ring theory (see [3, 4, 5, 6, 7, 8] and references therein). In particular, Barnes [2] and Kyuno [5, 6] studied the structure of Γ-rings and obtained various generalizations of the corresponding results of ring theory.

2 Prilimaniries

If M and Γ are additive abelian groups and there exists a mapping $(.,.,.): M \times \Gamma \times M \to M$ which satisfies the following conditions:

(i) (a,β,b) is an element of M,

(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$,

$a(\alpha+\beta)b = a\alpha b + a\beta b,$ *and*

$a\alpha(b+c) = a\alpha b + a\alpha c,$

(iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for $a,b,c \in M$ and α, $\beta \in \Gamma$, then M is called a Γ- ring [2].

It is known that from $(i)-(iii)$ the following result follows:

(*) $0\alpha b = a0b = a\alpha 0 = 0$ for all a and b in M and all α in Γ [2].

An additive mapping $*$ on a Γ-ring M is said to be an involution if $(x\gamma y)^* = y^*\gamma x^*$ and $(x^*)^* = x$ for all $x,y \in M, \gamma \in \Gamma$. A Γ-ring M is said to be commutative if $x\beta y = y\beta x$ for all $x,y \in M, \beta \in \Gamma$. A Γ-ring M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. Moreover, the set $Z(M) = \{x \in M : x\alpha y = y\alpha x \, \forall \alpha \in \Gamma, y \in M\}$ is called the centre of the Γ-ring M. We shall write $[x,y]_\alpha = x\alpha y - y\alpha x$, $x,y \in M$ and $\alpha \in \Gamma$. We shall make use of the basic commutator identities: $[x\alpha y, z]_\beta = [x,z]_\beta \alpha y + x[\alpha,\beta]_z y + x\alpha[y,z]_\beta$ and $[x, y\alpha z]_\beta = [x,y]_\beta \alpha z + y[\alpha,\beta]_x z + y\alpha[x,z]_\beta$, for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$. If a Γ-ring satisfies the assumption

(**) $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M, \alpha,\beta \in \Gamma$.

Then the previous identities reduce to $[x\beta y, z]_\alpha = [x,z]_\alpha \beta y + x\beta[y,z]_\alpha$ and $[x, y\beta z]_\alpha = [x,y]_\alpha \beta z + y\beta[x,z]_\alpha$, for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$.

An additive mapping $D: M \to M$ is called a * -derivation on M if $D(x\gamma y) = D(x)\gamma y^* + x\gamma D(y)$ for all $x,y \in M$ and $\gamma \in \Gamma$. An additive mapping $D: M \to M$ is called a Jordan * -derivation on M if $D(x\gamma x) = D(x)\gamma x^* + x\gamma D(x)$ for all $x \in M$ and $\gamma \in \Gamma$. A mapping F from M to M is said to be commuting on M if $[F(x),x]_\gamma = 0$ and centralizing on M if $[F(x),x]_\gamma \in Z(M)$ for all $x \in M, \gamma \in \Gamma$. An additive mapping $T: M \to M$ is said to be left (right) * -centralizer if $T(x\gamma y) = T(x)\gamma y^*$ ($T(x\gamma y) = x^*\gamma T(y)$) for all $x,y \in M, \gamma \in \Gamma$. A * -centralizer is an additive mapping which is both a left and a right * -centralizer. An additive mapping $T: M \to M$ is said to be Jordan left (right) * -centralizer if $T(x\gamma x) = T(x)\gamma x^*$ ($T(x\gamma x) = x^*\gamma T(x)$) for all $x \in M, \gamma \in \Gamma$.

Let R be a ring. It is known [10] that if a mapping $T: R \to R$ is both a left and a right Jordan centralizer then T satisfies $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$ but an additive mapping $T: R \to R$ satisfying $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$ need not be a left and a right Jordan centralizer. But in [10] he has also proved that if a 2-torsion free semiprime ring R admits an additive mapping T satisfying $2T(x^2) = T(x)x + xT(x)$ for all $x \in R,$ then T is a left and a right centralizer.

In this paper, motivated from the following Example 1.1, we generalize the identity in [10] for Jordan *-centralizers in semiprime Γ-rings with involution.

Example 1.1 *Let R and Z are commutative ring of real numbers and integers, respectively.*

$$M = M_{2,2}(R) = \left\{ \begin{pmatrix} m & 0 \\ n & k \end{pmatrix} : m, n, k \in R \right\}$$

Let

$$\Gamma = \Gamma_{2,2}(Z) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in Z \right\}$$

Then $M \times \Gamma \times M \to M$ is a Γ-ring under usual addition and multiplication of matrices.

The Γ-ring M is given by

$$(A, \chi, B) = A\chi B = \left\{ \begin{pmatrix} m_1\alpha m_2 & 0 \\ n_1\alpha m_2 + k_1\beta n_2 & k_1\beta k_2 \end{pmatrix} : \alpha, \beta \in Z, m_1, m_2, n_1, n_2, k_1, k_2 \in R \right\}$$

It is easy to verify that M is semiprime Γ-ring. We define an additive mapping $f: M \to M$ by

$$f\begin{pmatrix} m_1\alpha m_2 & 0 \\ n_1\alpha m_2 + k_1\beta n_2 & k_1\beta k_2 \end{pmatrix} = \begin{pmatrix} m_1\alpha m_2 & 0 \\ 0 & 0 \end{pmatrix}, \text{where } \alpha, \beta \in Z,$$
$$m_1, m_2, n_1, n_2, k_1, k_2 \in R.$$

Let $*: M \to M$ is an involution defined by

$$*\begin{pmatrix} m_1\alpha m_2 & 0 \\ n_1\alpha m_2 + k_1\beta n_2 & k_1\beta k_2 \end{pmatrix} = \begin{pmatrix} m_1\alpha m_2 & 0 \\ 0 & k_1\beta k_2 \end{pmatrix},$$
$$\text{where } \alpha, \beta \in Z,$$
$$m_1, m_2, n_1, n_2, k_1, k_2 \in R.$$

It is easy to check that

$2f(A\xi A) = f(A)\xi A^* + A^*\xi f(A)$ for all $A \in M$, $\xi \in \Gamma$. Then f is a Jordan *-centralizer.

3 On Jordan *-centralizers in semiprime Γ-rings with involution

In this section we prove our result regarding Jordan *-centralizers in semiprime Γ-rings with involution.

Lemma 2.1 *Let M be a 2-torsion free semiprime Γ-ring with involution satisfying $x\gamma y\alpha z = x\alpha y\gamma z$ and $f: M \to M$ an additive mapping such that $2f(x\beta x) = f(x)\beta x^* + x^*\beta f(x)$ for all $x \in M$, $\beta \in \Gamma$, then f satisfies the identity $[f(x), x^*\beta x^*]_\gamma = 0.$*

Proof. We assume that M is noncommutative (the theorem is trivial when M is commutative).

Linearizing

$$2f(x\beta x) = f(x)\beta x^* + x^*\beta f(x), \tag{1}$$

then using the last relation, we get

$$2f(x\beta y + y\beta x) = f(x)\beta y^* + x^*\beta f(y) + f(y)\beta x^* + y^*\beta f(x). \tag{2}$$

Replacing y by $2(x\gamma y + y\gamma x)$ in (2), we obtain

$$4f(x\beta(x\gamma y + y\gamma x) + (x\gamma y + y\gamma x)\beta x) = 2f(x)\beta(y^*\gamma x^* + x^*\gamma y^*) + 2x^*\beta f(x\gamma y + y\gamma x) +2f(x\gamma y + y\gamma x)\beta x^* + 2(y^*\gamma x^* + x^*\gamma y^*)\beta f(x).$$

Using (2), from the last relation we get

$$4f(x\beta(x\gamma y + y\gamma x) + (x\gamma y + y\gamma x)\beta x) = 2f(x)\beta(y^*\gamma x^* + x^*\gamma y^*) + x^*\beta f(x)\gamma y^* +x^*\beta x^*\gamma f(y) + 2x^*\beta f(y)\gamma x^* + x^*\beta y^*\gamma f(x) + f(x)\gamma y^*\beta x^* + f(y)\gamma x^*\beta x^* +y^*\gamma f(x)\beta x^* + 2(y^*\gamma x^* + x^*\gamma y^*)\beta f(x).$$

That is,

$$4f(x\beta(x\gamma y + y\gamma x) + (x\gamma y + y\gamma x)\beta x) = f(x)\beta(2x^*\gamma y^* + 3y^*\gamma x^*) +(3x^*\gamma y^* + 2y^*\gamma x^*)\beta f(x) + x^*\beta f(x)\gamma y^* +y^*\gamma f(x)\beta x^* + 2x^*\beta f(y)\gamma x^* + x^*\beta x^*\gamma f(y) + f(y)\gamma x^*\beta x^*, \tag{3}$$

which alongwith (1) and (2) gives

$$4f(x\beta(x\gamma y + y\gamma x) + (x\gamma y + y\gamma x)\beta x) = f(x)\beta x^*\gamma y^* +y^*\gamma x^*\beta f(x) + x^*\beta f(x)\gamma y^* + y^*\gamma f(x)\beta x^* +2x^*\beta x^*\gamma f(y) + 2f(y)\gamma x^*\beta x^* + 8f(x\gamma y\beta x). \tag{4}$$

Comparing (3) and (4), we get

$$8f(x\gamma y\beta x) = f(x)\beta(x^*\gamma y^* + 3y^*\gamma x^*) + (3x^*\gamma y^* + y^*\gamma x^*)\beta f(x) + 2x^*\beta f(y)\gamma x^* - x^*\beta x^*\gamma f(y) - f(y)\gamma x^*\beta x^*. \tag{5}$$

Replacing y by $8(x\gamma y\beta x)$ in (2), we get

$$16f(x\beta(x\gamma y\beta x)+(x\gamma y\beta x)\beta x)=8f(x)\beta(x^*\gamma y^*\beta x^*)$$
$$+8x^*\beta f(x\gamma y\beta x)+8f(x\gamma y\beta x)\beta x^*$$
$$+8(x^*\gamma y^*\beta x^*)\beta f(x).$$

Using (5), from the last relation we get

$$16f(x\beta(x\gamma y\beta x)+(x\gamma y\beta x)\beta x)=f(x)\beta(9x^*\gamma y^*\beta x^*+3y^*\gamma x^*\beta x^*)$$
$$+(9x^*\gamma y^*\beta x^*+3x^*\beta x^*\gamma y^*)\beta f(x)+x^*\beta f(x)\beta(x^*\gamma y^*+3y^*\gamma x^*)$$
$$+(3x^*\gamma y^*+y^*\gamma x^*)\beta f(x)\beta x^*+(x^*\beta)^2 f(y)\gamma x^*$$
$$+x^*\gamma f(y)(\beta x^*)^2-f(y)\gamma x^*(\beta x^*)^2-(x^*\beta)^2 x^*\gamma f(y), \qquad (6)$$

which alongwith (5) gives

$$16f(x\beta(x\gamma y\beta x)+(x\gamma y\beta x)\beta x)=$$
$$2[8f(x\gamma(x\beta y)\beta x)]+2[8f(x)\gamma(y\beta x)\beta x]$$
$$=f(x)\beta(2x^*\beta x^*\gamma y^*+6y^*\gamma x^*\beta x^*$$
$$+8x^*\gamma y^*\beta x^*)+(6x^*\beta x^*\gamma y^*+2y^*\gamma x^*\beta x^*$$
$$+8x^*\gamma y^*\beta x^*)\beta f(x)+4x^*\beta f(x\beta y+y\beta x)\gamma x^*$$
$$-2x^*\beta x^*\gamma f(x\beta y+y\beta x)-2f(x\beta y+y\beta x)\gamma x^*\beta x^*.$$

Using (2), from the last relation we get

$$16f(x\beta(x\gamma y\beta x)+(x\gamma y\beta x)\beta x)=$$
$$f(x)\beta(2x^*\beta x^*\gamma y^*+5y^*\gamma x^*\beta x^*+8x^*\gamma y^*\beta x^*)$$
$$+(2y^*\gamma x^*\beta x^*+5x^*\beta x^*\gamma y^*+8x^*\gamma y^*\beta x^*)\beta f(x)$$
$$+2x^*\beta f(x)\gamma y^*\beta x^*+2x^*\beta y^*\gamma f(x)\beta x^*$$
$$+(x^*\beta)^2 f(y)\gamma x^*+x^*\gamma f(y)(\beta x^*)^2$$
$$-(x^*\beta)^2 f(x)\gamma y^*-y^*\gamma f(x)(\beta x^*)^2$$
$$-(x^*\beta)^2 x^*\gamma f(y)-f(y)\gamma x^*(\beta x^*)^2. \qquad (7)$$

Comparing (6) and (7), we get

$$f(x)\beta(x^*\gamma y^*\beta x^*-2y^*\gamma x^*\beta x^*-2x^*\beta x^*\gamma y^*)$$
$$+(x^*\gamma y^*\beta x^*-2x^*\beta x^*\gamma y^*-2y^*\gamma x^*\beta x^*)\beta f(x)$$
$$+x^*\beta f(x)\beta(x^*\gamma y^*+y^*\gamma x^*)+(x^*\gamma y^*+y^*\gamma x^*)\beta f(x)\beta x^*$$
$$+(x^*\beta)^2 f(x)\gamma y^*+y^*\gamma f(x)(\beta x^*)^2=0. \qquad (8)$$

Replacing y by $x\beta y$ in the last relation, we get

$$f(x)\beta(x^*\gamma y^*(\beta x^*)^2-2y^*\gamma x^*(\beta x^*)^2-2(x^*\beta)^2 y^*\gamma x^*)$$
$$+(x^*\gamma y^*(\beta x^*)^2-2(x^*\beta)^2 y^*\gamma x^*-2y^*\gamma x^*(\beta x^*)^2)\beta f(x)$$
$$+x^*\beta f(x)\beta(x^*\beta y^*\gamma x^*+y^*\gamma x^*\beta x^*)+(x^*\beta)^2 f(x)\beta y^*\gamma x^*$$
$$+(x^*\beta y^*\gamma x^*+y^*\gamma x^*\beta x^*)\beta f(x)\beta x^*$$
$$+y^*\gamma x^*\beta f(x)(\beta x^*)^2=0. \qquad (9)$$

Equation (8) alongwith (*) implies

$$f(x)\beta(x^*\gamma y^*\beta x^*-2y^*\gamma x^*\beta x^*-2x^*\beta x^*\gamma y^*)\beta x^*+$$
$$(x^*\gamma y^*\beta x^*-2x^*\beta x^*\gamma y^*-2y^*\gamma x^*\beta x^*)\beta f(x)\beta x^*$$
$$+x^*\beta f(x)\beta(x^*\gamma y^*+y^*\gamma x^*)\beta x^*$$
$$+(x^*\gamma y^*+y^*\gamma x^*)\beta f(x)(\beta x^*)^2+(x^*\beta)^2 f(x)\gamma y^*\beta x^*$$
$$+y^*\gamma f(x)(\beta x^*)^3=0. \qquad (10)$$

Subtracting (10) from (9), we have

$$x^*\beta y^*\beta x^*\beta[x^*,f(x)]_\gamma+x^*\beta y^*\beta[x^*,f(x)]_\gamma\beta x^*$$
$$+2(x^*\beta)^2 y^*\beta[f(x),x^*]_\gamma+2y^*\beta(x^*\beta)^2[f(x),x^*]_\gamma$$
$$+y^*\beta x^*\beta[x^*,f(x)]_\gamma\beta x^*+y^*\beta[x^*,f(x)]_\gamma(\beta x^*)^2=0.$$

That is,

$$x^*\beta y^*\beta[x^*\beta x^*,f(x)]_\gamma+2(x^*\beta)^2 y^*\beta[f(x),x^*]_\gamma$$
$$+2y^*\beta(x^*\beta)^2[f(x),x^*]_\gamma+y^*\beta x^*\beta[x^*,f(x)]_\gamma\beta x^*$$
$$+y^*\beta[x^*,f(x)]_\gamma(\beta x^*)^2=0. \qquad (11)$$

Replacing y^* by $f(x)\gamma y^*$ in the last relation, we get

$$x^*\beta f(x)\gamma y^*\beta[x^*\beta x^*,f(x)]_\gamma+2(x^*\beta)^2 f(x)\gamma y^*$$
$$\beta[f(x),x^*]_\gamma+2f(x)\gamma y^*\beta(x^*\beta)^2[f(x),x^*]_\gamma$$
$$+f(x)\gamma y^*\beta x^*\beta[x^*,f(x)]_\gamma\beta x^*+$$
$$f(x)\gamma y^*\beta[x^*,f(x)]_\gamma(\beta x^*)^2=0. \qquad (12)$$

Equation (11) alongwith (*) implies

$$f(x)\gamma x^*\beta y^*\beta[x^*\beta x^*,f(x)]_\gamma+2f(x)\gamma(x^*\beta)^2 y^*$$
$$\beta[f(x),x^*]_\gamma+2f(x)\gamma y^*\beta(x^*\beta)^2[f(x),x^*]_\gamma$$
$$+f(x)\gamma y^*\beta x^*\beta[x^*,f(x)]_\gamma\beta x^*$$
$$+f(x)\gamma y^*\beta[x^*,f(x)]_\gamma(\beta x^*)^2=0. \qquad (13)$$

Subtracting (13) from (12) we get

$$[f(x),x^*]_\beta\gamma y^*\beta[f(x),x^*\beta x^*]_\gamma$$
$$-2[f(x),x^*\beta x^*]_\gamma\beta y^*\gamma[f(x),x^*]_\beta=0. \qquad (14)$$

Replacing y^* by y. Let $a=[f(x),x^*]_\beta$, $b=[f(x),x^*\beta x^*]_\gamma$ and $c=-2[f(x),x^*\beta x^*]_\gamma$.

Then equation (14) becomes

$$a\gamma y\beta b+c\beta y\gamma a=0$$
$$\text{for all } y\in M,\ \beta,\gamma\in\Gamma. \qquad (15)$$

Let $z\in M$. Replacing y by $y\gamma a\beta z$ in the last relation we get

$$a\gamma y\gamma a\beta z\beta b+c\beta y\gamma a\beta z\gamma a=0$$
$$\text{for all } y,z\in M,\ \beta,\gamma\in\Gamma. \qquad (16)$$

Equation (15) alongwith (*) implies

$$a\gamma y\beta a\gamma z\beta b + a\gamma y\beta c\beta z\gamma a = 0$$

$$\text{for all } y \in M,\ \beta,\gamma \in \Gamma. \qquad (17)$$

Subtracting (16) from (17) we obtain

$$(a\gamma y\beta c - c\beta y\gamma a)\beta z\gamma a = 0$$

$$\text{for all } y,z \in M,\ \beta,\gamma \in \Gamma. \qquad (18)$$

Replacing z by $z\gamma c\beta y$ in the last relation we get

$$(a\gamma y\beta c - c\beta y\gamma a)\beta z\gamma c\beta y\gamma a = 0. \qquad (19)$$

Equation (18) alongwith $(*)$ gives

$$(a\gamma y\beta c - c\beta y\gamma a)\beta z\gamma a\gamma y\beta c = 0. \qquad (20)$$

Subtracting (19) from (20) we obtain

$$(a\gamma y\beta c - c\beta y\gamma a)\beta z\gamma(a\gamma y\beta c - c\beta y\gamma a) = 0, \quad (21)$$

which alongwith semiprimeness of M implies

$$a\gamma y\beta c = c\beta y\gamma a \qquad (22)$$

Using (22), from (15) we get $a\gamma y\beta(b+c) = 0.$

In other words

$$[f(x),x^*]_\beta \gamma y\beta[f(x),x^*\beta x^*]_\gamma = 0. \qquad (23)$$

The last relation gives

$$([f(x),x^*]_\beta\gamma x^* + x^*\gamma[f(x),x^*]_\beta)\gamma y\beta[f(x),x^*\beta x^*]_\gamma = 0,$$

Which implies

$$[f(x),x^*\gamma x^*]_\beta\gamma y\beta[f(x),x^*\beta x^*]_\beta = 0.$$

That is,

$$[f(x),x^*\beta x^*]_\gamma\gamma y\beta[f(x),x^*\beta x^*]_\gamma = 0.$$

Semiprimeness of M implies

$$[f(x),x^*\beta x^*]_\gamma = 0 \qquad (24)$$

Theorem 2.2 *Let M be a 2-torsion free semiprime Γ-ring with involution satisfying $x\gamma y\alpha z = x\alpha y\gamma z$ and $f : M \to M$ an additive mapping such that*

$$2f(x\beta x) = f(x)\beta x^* + x^*\beta f(x)$$

for all $x \in M$, $\beta \in \Gamma$, then f is a Jordan *-centralizer.

Proof. By Lemma 2.1, linearizing (24), we get

$$[f(x),y^*\beta y^*]_\gamma + [f(y),x^*\beta x^*]_\gamma + [f(x),x^*\beta y^* + y^*\beta x^*]_\gamma$$

$$+[f(y),x^*\beta y^* + y^*\beta x^*]_\gamma = 0.$$

Replacing x by $-x$ in the last relation and comparing the relation so obtained with the last relation alongwith 2-torsionfreeness of M, we get

$$[f(x),x^*\beta y^* + y^*\beta x^*]_\gamma + [f(y),x^*\beta x^*]_\gamma = 0 \qquad (25)$$

Replacing y by $2(y\beta x + x\beta y)$ in the last relation and then using (2) and (24), we get

$$0 = 2[f(x),(x^*\beta)^2 y^* + y^*(\beta x^*)^2 + 2x^*\beta y^*\beta x^*]_\gamma$$

$$+[f(y)\beta x^* + y^*\beta f(x) + f(x)\beta y^* + x^*\beta f(y), x^*\beta x^*]_\gamma$$

$$= 2(x^*\beta)^2[f(x),y^*]_\gamma + 2[f(x),y^*]_\gamma(\beta x^*)^2$$

$$+4[f(x),x^*\beta y^*\beta x^*]_\gamma + f(x)\beta[y^*,x^*\beta x^*]_\gamma$$

$$+x^*\beta[f(y),x^*\beta x^*]_\gamma + [f(y),x^*\beta x^*]_\gamma\beta x^*$$

$$+[y^*,x^*\beta x^*]_\gamma\beta f(x).$$

That is,

$$2(x^*\beta)^2[f(x),y^*]_\gamma + 2[f(x),y^*]_\gamma(\beta x^*)^2$$

$$+4[f(x),x^*\beta y^*\beta x^*]_\gamma + f(x)\beta[y^*,x^*\beta x^*]_\gamma$$

$$+x^*\beta[f(y),x^*\beta x^*]_\gamma + [f(y),x^*\beta x^*]_\gamma\beta x^*$$

$$+[y^*,x^*\beta x^*]_\gamma\beta f(x) = 0 \qquad (26)$$

Replacing y by x in the last relation and using 2-torsionfreeness of M, we obtain

$$(x^*\beta)^2[f(x),x^*]_\gamma + [f(x),x^*]_\gamma(\beta x^*)^2$$

$$+2[f(x),(x^*\beta)^2 x^*]_\gamma = 0,$$

which gives

$$(x^*\beta)^2[f(x),x^*]_\gamma + 3[f(x),x^*]_\gamma(\beta x^*)^2 = 0 \quad (27)$$

From (24) we get

$$[f(x),x^*]_\gamma\beta x^* + x^*\beta[f(x),x^*]_\gamma = 0 \qquad (28)$$

From the last relation by easy calculations one gets $(x^*\beta)^2[f(x),x^*]_\gamma = [f(x),x^*]_\gamma(\beta x^*)^2.$ Using the last relation, from (27) alongwith 2-torsionfreeness of M, we get

$$[f(x),x^*]_\gamma(\beta x^*)^2 = 0 \qquad (29)$$

and

$$(x^*\beta)^2[f(x),x^*]_\gamma = 0 \qquad (30)$$

From (28) and then using (30) we attain

$$x^*\beta[f(x),x^*]_\gamma\beta x^* = 0 \qquad (31)$$

Using (25), from (26) we have

$$0 = 2(x^*\beta)^2[f(x), y^*]_\gamma + 2[f(x), y^*]_\gamma(\beta x^*)^2$$
$$+4[f(x), x^*\beta y^*\beta x^*]_\gamma + f(x)\beta[y^*, x^*\beta x^*]_\gamma$$
$$+[y^*, x^*\beta x^*]_\gamma \beta f(x) - x^*\beta[f(x), x^*\beta y^* + y^*\beta x^*]_\gamma$$
$$-[f(x), x^*\beta y^* + y^*\beta x^*]_\gamma \beta x^*$$
$$= 2(x^*\beta)^2[f(x), y^*]_\gamma + 2[f(x), y^*]_\gamma(\beta x^*)^2$$
$$+4[f(x), x^*\beta y^*\beta x^*]_\gamma + f(x)\beta[y^*, x^*\beta x^*]_\gamma$$
$$+[y^*, x^*\beta x^*]_\gamma \beta f(x) - x^*\beta[f(x), x^*]_\gamma \beta y^*$$
$$-(x^*\beta)^2[f(x), y^*]_\gamma - x^*\beta[f(x), y^*]_\gamma$$
$$-x^*\beta y^*\beta[f(x), x^*]_\gamma - [f(x), x^*]_\gamma \beta y^*\beta x^*$$
$$-x^*\beta[f(x), y^*]_\gamma \beta x^* - [f(x), y^*]_\gamma(\beta x^*)^2$$
$$-y^*\beta[f(x), x^*]_\gamma \beta x^*.$$

That is,

$$(x^*\beta)^2[f(x), y^*]_\gamma + [f(x), y^*]_\gamma(\beta x^*)^2$$
$$+3[f(x), x^*]_\gamma \beta y^*\beta x^* + 3x^*\beta y^*\beta[f(x), x^*]_\gamma$$
$$+2x^*\beta[f(x), y^*]_\gamma \beta x^* + f(x)\beta[y^*, x^*\beta x^*]_\gamma$$
$$+[y^*, x^*\beta x^*]_\gamma \beta f(x) - x^*\beta[f(x), x^*]_\gamma \beta y^*$$
$$-y^*\beta[f(x), x^*]_\gamma \beta x^* = 0 \qquad (32)$$

Replacing y by $x\beta y$ in the last relation we obtain

$$(x^*\beta)^2[f(x), y^*]_\gamma \beta x^* + (x^*\beta)^2 y^*\beta[f(x), x^*]_\gamma$$
$$+[f(x), y^*]_\gamma \beta(x^*\beta)^2 x^* + y^*\beta[f(x), x^*]_\gamma(\beta x^*)^2$$
$$+3[f(x), x^*]_\gamma \beta y^*(\beta x^*)^2 + 3x^*\beta y^*\beta x^*\beta[f(x), x^*]_\gamma$$
$$+2x^*\beta[f(x), y^*]_\gamma(\beta x^*)^2 + 2x^*\beta y^*\beta[f(x), x^*]_\gamma \beta x^*$$
$$-x^*\beta[f(x), x^*]_\gamma \beta y^*\beta x^* + f(x)\beta[y^*, x^*\beta x^*]_\gamma \beta x^*$$
$$+[y^*, x^*\beta x^*]_\gamma \beta x^*\beta f(x) - y^*\beta x^*\beta[f(x), x^*]_\gamma \beta x^* = 0.$$

Using (29) and (30), the last relation reduces to

$$(x^*\beta)^2[f(x), y^*]_\gamma \beta x^* + (x^*\beta)^2 y^*\beta[f(x), x^*]_\gamma$$
$$+[f(x), y^*]_\gamma \beta(x^*\beta)^2 x^* + 3[f(x), x^*]_\gamma \beta y^*(\beta x^*)^2$$
$$+3x^*\beta y^*\beta x^*\beta[f(x), x^*]_\gamma + 2x^*\beta[f(x), y^*]_\gamma(\beta x^*)^2$$
$$+2x^*\beta y^*\beta[f(x), x^*]_\gamma \beta x^* - x^*\beta[f(x), x^*]_\gamma \beta y^*\beta x^*$$
$$+f(x)\beta[y^*, x^*\beta x^*]_\gamma \beta x^* + [y^*, x^*\beta x^*]_\gamma \beta x^*\beta f(x) = 0. \qquad (33)$$

Equation (32) alongwith $(*)$ implies

$$(x^*\beta)^2[f(x), y^*]_\gamma \beta x^* + [f(x), y^*]_\gamma(\beta x^*)^3$$
$$+3[f(x), x^*]_\gamma \beta y^*(\beta x^*)^2 + 3x^*\beta y^*\beta[f(x), x^*]_\gamma \beta x^*$$
$$+2x^*\beta[f(x), y^*]_\gamma(\beta x^*)^2 + f(x)\beta[y^*, x^*\beta x^*]_\gamma \beta x^*$$
$$+[y^*, x^*\beta x^*]_\gamma \beta f(x)\beta x^* - x^*\beta[f(x), x^*]_\gamma \beta y^*\beta x^*$$
$$-y^*\beta[f(x), x^*]_\gamma(\beta x^*)^2 = 0 \qquad (34)$$

Subtracting (34) from (33), we have

$$(x^*\beta)^2 y^*\beta[f(x), x^*]_\gamma + 3x^*\beta y^*\beta x^*\beta[x^*, [f(x), x^*]_\gamma]_\gamma$$
$$+2x^*\beta y^*\beta[f(x), x^*]_\gamma \beta x^* + [y^*, x^*\beta x^*]_\gamma \beta[x^*, f(x)]_\gamma = 0,$$

which alongwith (30) gives

$$2(x^*\beta)^2 y^*\beta[f(x), x^*]_\gamma + 3x^*\beta y^*\beta x^*\beta[f(x), x^*]_\gamma$$
$$-x^*\beta y^*\beta[f(x), x^*]_\gamma \beta x^* = 0.$$

Replacing $-[f(x), x^*]_\gamma \beta x^*$ by $x^*\beta[f(x), x^*]_\gamma$ in the last relation, we obtain

$$(x^*\beta)^2 y^*\beta[f(x), x^*]_\gamma + 2x^*\beta y^*\beta x^*\beta[f(x), x^*]_\gamma = 0 \qquad (35)$$

Using (24), (29), (30) and (31), from (11) we get

$$(x^*\beta)^2 y^*\beta[f(x), x^*]_\gamma = 0,$$

which alongwith (35), gives $x^*\beta y^*\beta x^*\beta[f(x), x^*]_\gamma = 0$.

That is,

$$x^*\beta[f(x), x^*]_\gamma \gamma y^*\beta x^*\beta[f(x), x^*]_\gamma = 0.$$

Replacing y^* by y alongwith semiprimeness of M, we have

$$x^*\beta[f(x), x^*]_\gamma = 0 \qquad (36)$$

Similarly, we have

$$[f(x), x^*]_\gamma \beta x^* = 0 \qquad (37)$$

Linearizing (36) and then replacing x by $-x$ and adding the relation so obtained in the previous relation and then using 2-torsionfreeness of M, we get

$$y^*\beta[f(x), x^*]_\gamma + x^*\beta[f(x), y^*]_\gamma + x^*\beta[f(y), x^*]_\gamma = 0$$

The last relation alongwith (37) and $(*)$, implies

$$[f(x), x^*]_\gamma \beta y^*\gamma[f(x), x^*]_\gamma = 0.$$

Replacing y^* by y alongwith semiprimeness of M, we get

$$[f(x), x^*]_\gamma = 0 \qquad (38)$$

Combining (1) and (38), we get $f(x\beta x) = f(x)\beta x^*$ and $f(x\beta x) = x^*\beta f(x)$ for all $x \in M$ and $\beta \in \Gamma$. That is, f is both a left and a right Jordan *-centralizer. Hence f is a Jordan *-centralizer.

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