ON JORDAN * CENTRALIZERS IN SEMIPRIME GAMMA-RINGS WITH INVOLUTION

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ABSTRACT: In this paper we prove that if \( M \) is a 2-torsion free semiprime \( \Gamma \)-ring with involution satisfying \( x\gamma y = x\alpha y \gamma \) and \( f: M \rightarrow M \) an additive mapping such that

\[ 2f(x\beta y) = f(x)\beta x^* + x^* \beta f(x) \]

for all \( x, \beta \in M, \beta \in \Gamma \), then \( f \) is a Jordan *-centralizer.

Keywords: Semiprime \( \Gamma \)-ring, involution, *-derivation, Jordan *-derivation, left (right) Jordan *-centralizer, \( \Gamma \)-centralizer, commutators.

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1 INTRODUCTION

Nobusawa [1] introduced the notion of a \( \Gamma \)-ring, a notion more general than a ring. Barnes [2] slightly weakened the conditions in the definition of a \( \Gamma \)-ring given by Nobusawa. After the study of \( \Gamma \)-rings by Nobusawa [1] and Barnes [2] many researchers have done a lot of work on \( \Gamma \)-rings and have obtained some generalizations of the corresponding results in ring theory (see [3, 4, 5, 6, 7, 8] and references therein). In particular, Barnes [2] and Kyuno [5, 6] studied the structure of \( \Gamma \)-rings and obtained various generalizations of the corresponding results of ring theory.

2 Preliminaries

If \( M \) and \( \Gamma \) are additive abelian groups and there exists a mapping \( (., ., .) : M \times \Gamma \times M \rightarrow M \) which satisfies the following conditions:

(i) \( (a, \beta, b) \) is an element of \( M \),

(ii) \( (a + b)\alpha c = a\alpha b + b\alpha c \),

\[ a(\alpha + \beta)b = a\alpha b + a\beta b, \quad \text{and} \]

\[ a\alpha(b + c) = a\alpha b + a\alpha c, \]

(iii) \( (a\alpha b)\beta c = a\alpha(b\beta c) \), for \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \), then \( M \) is called a \( \Gamma \)-ring [2].

It is known that from (i) – (iii) the following result follows:

(*') \( 0\alpha b = a0b = a\alpha 0 = 0 \) for all \( a \) and \( b \) in \( M \) and all \( \alpha \) in \( \Gamma \) [2].

An additive mapping \( * \) on a \( \Gamma \)-ring \( M \) is said to be an involution if \( (x\gamma y)^* = y^* \alpha x^* \) and \( (x^*)^* = x \) for all \( x, y \in M, \gamma \in \Gamma \). A \( \Gamma \)-ring \( M \) is said to be commutative if \( x\beta y = y\beta x \) for all \( x, y \in M, \beta \in \Gamma \). A \( \Gamma \)-ring \( M \) is said to be 2-torsion free if \( 2x = 0 \) implies \( x = 0 \) for all \( x \in M \). Moreover, the set \( Z(M) = \{x \in M : x\alpha y = y\alpha x \forall \alpha \in \Gamma, y \in M \} \) is called the centre of the \( \Gamma \)-ring \( M \). We shall write \( [x, y] = x\alpha y - y\alpha x, x, y \in M \) and \( \alpha \in \Gamma \). We shall make use of the basic commutator identities:

\[ [x\alpha y, z] = [x, z]_\beta \alpha y + [x, \alpha y]_\beta z + [x, \alpha z]_{\beta y} \]

and

\[ [x, y\alpha z] = [x, y]_\beta \alpha z + [y, \alpha z]_\beta x + [y\alpha x, [x, z]_\beta, \quad \text{for all} \] \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \). If a \( \Gamma \)-ring satisfies the assumption

\[ (**) \quad a\alpha b\beta c = a\beta b\alpha c \]

for all \( a, b, c \in M, \alpha, \beta \in \Gamma \). Then the previous identities reduce to

\[ [x\beta y, z] = [x, z]_\beta \alpha y + x\beta [y, z], \]

and

\[ [x, y\beta z] = [x, y]_\alpha \beta z + y\beta [x, z], \quad \text{for all} \]

\( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \). An additive mapping \( D: M \rightarrow M \) is called a \( \gamma \)-derivation on \( M \) if \( D(x\gamma y) = D(x)\gamma y^* + x\gamma D(y) \) for all \( x, y \in M \) and \( \gamma \in \Gamma \). An additive mapping \( D: M \rightarrow M \) is called a Jordan *-derivation on \( M \) if \( D(x\gamma x) = D(x\gamma x^*) + x\gamma D(x) \) for all \( x \in M \) and \( \gamma \in \Gamma \). A mapping \( F \) from \( M \) to \( M \) is said to be commuting on \( M \) if \( [F(x), x] = 0 \) and centralizing on \( M \) if \( [F(x), x] \in Z(M) \) for all \( x \in M, \gamma \in \Gamma \). An additive mapping \( T: M \rightarrow M \) is said to be left (right) *-centralizer if \( T(x\gamma y) = T(x)\gamma y^* \) (\( T(x\gamma y) = x^* \gamma T(y) \)) for all \( x, y \in M, \gamma \in \Gamma \). A *-centralizer is an additive mapping which is both a left and a right *-centralizer. An additive mapping \( T: M \rightarrow M \) is said to be Jordan left (right) *-centralizer if \( T(x\gamma x) = T(x)\gamma x^* \) (\( T(x\gamma x) = x^* \gamma T(x) \)) for all \( x \in M, \gamma \in \Gamma \).

Let \( R \) be a ring. It is known [10] that if a mapping \( T: R \rightarrow R \) is both a left and a right Jordan centralizer then \( T \) satisfies \( 2T(x^2) = T(x)x + xT(x) \) for all \( x \in R \) but
an additive mapping \( T : R \to R \) satisfying 
\[ 2T(x^2) = T(x)x + xT(x) \] for all \( x \in R \) need not be a left and a right Jordan centralizer. But in [10] he has also
proved that if a 2-torsion free semiprime ring \( R \) admits an
additive mapping \( T \) satisfying 
\[ 2T(x^2) = T(x)x + xT(x) \] for all \( x \in R \), then \( T \) is a
left and a right centralizer.

In this paper, motivated from the following Example 1.1, we
generalize the identity in [10] for Jordan \(*\)-centralizers in semiprime \( \Gamma \)-rings with involution.

**Example 1.1** Let \( R \) and \( Z \) are commutative ring of real
numbers and integers, respectively.
\[
M = M_{2,2}(R) = \left\{ \begin{pmatrix} m & 0 \\ n & k \end{pmatrix} : m,n,k \in R \right\}
\]
Let
\[
\Gamma = \Gamma_{2,2}(Z) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha,\beta \in Z \right\}
\]
Then \( M \times \Gamma \times M \to M \) is a \( \Gamma \)-ring under usual addition
and multiplication of matrices.
The \( \Gamma \)-ring \( M \) is given by
\[
(A, \chi, B) = A\chi B = \begin{pmatrix}
m_{1,am_2} & 0 \\
n_{1,am_2} + k_1\beta n_2 & k_1\beta k_2
\end{pmatrix}; \quad \alpha,\beta \in Z, m_1,m_2,n_1,n_2,k_1,k_2 \in R
\]
It is easy to verify that \( M \) is semiprime \( \Gamma \)-ring. We define
an additive mapping \( f : M \to M \) by
\[
f = \begin{pmatrix}
m_{1,am_2} & 0 \\
n_{1,am_2} + k_1\beta n_2 & k_1\beta k_2
\end{pmatrix}; \quad \text{where } \alpha,\beta \in Z,
\]
\[
m_1,m_2,n_1,n_2,k_1,k_2 \in R.
\]
Let \(* : M \to M \) is an involution defined by
\[
* = \begin{pmatrix}
m_{1,am_2} & 0 \\
n_{1,am_2} + k_1\beta n_2 & k_1\beta k_2
\end{pmatrix}; \quad \alpha,\beta \in Z,
\]
\[
m_1,m_2,n_1,n_2,k_1,k_2 \in R.
\]
It is easy to check that
\[
2f(A \otimes A) = f(A) \otimes A + A^* \otimes f(A) \quad \text{for all}
\]
\( A \in M, \xi \in \Gamma \). Then \( f \) is a Jordan \(*\)-centralizer.

3 On Jordan \(*\)-centralizers in semiprime \( \Gamma \)-rings with
involution
In this section we prove our result regarding Jordan \(*\)-
centralizers in semiprime \( \Gamma \)-rings with involution.

**Lemma 2.1** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring
with involution satisfying \( x \gamma y A z = x A y A y z \) and
\( f : M \to M \) an additive mapping such that
\[
2f(xA\xi) = f(x)A\xi + x^*A^\xi f(x) \quad \text{for all } x \in M, \beta \in \Gamma,
\]
then \( f \) satisfies the identity \( [f(x), x^*A^\xi]_\gamma = 0 \).

**Proof.** We assume that \( M \) is noncommutative (the theorem is trivial when \( M \) is commutative).
Linearizing
\[
2f(xA\xi) = f(x)A\xi + x^*A^\xi f(x),
\]
then using the last relation, we get
\[
2f(xA\xi) = f(x)A\xi + x^*A^\xi f(x) + f(y)A\xi f(x)
\]
Replacing \( y \) by \( 2(xy^* + yxy) \) in (2) , we obtain
\[
4f(xA\xi) = f(x)A\xi + x^*A^\xi f(x) + f(y)A\xi f(x) + f(z)A\xi f(x)
\]
Using (2), from the last relation we get
\[
4f(xA\xi) = f(x)A\xi + x^*A^\xi f(x) + f(y)A\xi f(x) + f(z)A\xi f(x)
\]
That is,
\[
4f(xA\xi) = f(x)A\xi + x^*A^\xi f(x) + f(y)A\xi f(x) + f(z)A\xi f(x)
\]
which along with (1) and (2) gives
\[
4f(xA\xi) = f(x)A\xi + x^*A^\xi f(x) + f(y)A\xi f(x) + f(z)A\xi f(x)
\]
Comparing (3) and (4), we get
\[
8f(xA\xi) = f(x)A\xi + x^*A^\xi f(x) + f(y)A\xi f(x) + f(z)A\xi f(x)
\]
Comparing (3) and (4), we get
\[
8f(xA\xi) = f(x)A\xi + x^*A^\xi f(x) + f(y)A\xi f(x) + f(z)A\xi f(x)
Replacing $y$ by $8(xy\beta x)$ in (2), we get
\[ 16 f(x(\beta y x y x) + (xy y x) = 8 f(x) \beta (x^2 y x y x^2) \]
\[+8x^2 \beta f(x) y x y x^2 + 8 f(x) \beta x y x^2 \]
\[+8(x^2 y x y x^2) \beta f(x). \]

Using (5), from the last relation we get
\[ 16 f(x(\beta y x y x) + (xy y x) = f(x) \beta (9x^2 y x y x^2 + 3y^2 x y x^2) \]
\[ + (9x^2 y x y x^2 + 3x^2 y x y x^2) f(x) + x^2 f(x) \beta x y x^2 + 3 y^2 x y x^2 \]
\[ + (3x^2 y x y x^2 + y^2 x y x^2) f(x) x y x^2 + (x^2 y x y x^2) + 2x^2 f(x) \beta x y x^2 \]
\[+ x^2 f(x) \beta x y x^2 + 3 x^2 f(x) \beta x y x^2. \]

Using (2), from the last relation we get
\[ 16 f(x(\beta y x y x) + (xy y x) = f(x) \beta (2x^2 y x y^2 + 6y^2 x y x^2) \]
\[ + 8x^2 y x y x^2 + (6x^2 y x y x^2 + 2y^2 x y x^2) \]
\[ + 8x^2 y x y x^2 \beta f(x) + 4x^2 f(x) \beta y x y x^2 \]
\[ - 2x^2 \beta x^2 y^2 f(x) + 2f(x) \beta y x y x^2 \]
\[ - 2f(x) \beta x^2 y^2 - 2f(x) \beta (x^2 y x y x^2). \]

Replacing $y$ by $f(x) y x y^2$ in the last relation, we get
\[ x^2 \beta y f(x) x^2 y^2 \beta [x^2 y^2 + f(x)] + 2 (x^2 y^2) f(x) \gamma y x^2 \]
\[ + 2(x^2 y^2) y^2 [f(x), x^2] + 2y^2 [x^2 y^2] [f(x), x^2] + \gamma y^2 [x^2 y^2, f(x), x^2] \]
\[ + y^2 [x^2 y^2, f(x), x^2] + y^2 [x^2 y^2, f(x), x^2] \]
\[ + y^2 [x^2 y^2, f(x), x^2] = 0. \]

Replacing $y^2$ by $f(x) y x y^2$ in the last relation, we get
\[ x^2 \beta y f(x) x^2 y^2 \beta [x^2 y^2 + f(x)] + 2 (x^2 y^2) f(x) \gamma y x^2 \]
\[ + 2(x^2 y^2) y^2 [f(x), x^2] + 2y^2 [x^2 y^2] [f(x), x^2] + \gamma y^2 [x^2 y^2, f(x), x^2] \]
\[ + y^2 [x^2 y^2, f(x), x^2] + y^2 [x^2 y^2, f(x), x^2] \]
\[ + y^2 [x^2 y^2, f(x), x^2] = 0. \]
\[ a\gamma y\beta y\gamma z\beta b + a\gamma y\beta c\beta z\gamma a = 0 \]
\[ \text{for all } y \in M, \; \beta, \gamma \in \Gamma. \]  
(17)

Subtracting (16) from (17) we obtain
\[ (a\gamma y\beta c - c\beta y\gamma a)\beta z\gamma a = 0 \]
\[ \text{for all } y, z \in M, \; \beta, \gamma \in \Gamma. \]  
(18)

Replacing \( z \) by \( z\gamma c \beta y \) in the last relation we get
\[ (a\gamma y\beta c - c\beta y\gamma a)\beta z\gamma a = 0. \]  
(19)

Equation (18) alongwith (17) gives
\[ (a\gamma y\beta c - c\beta y\gamma a)\beta z\gamma a\gamma y\beta c = 0. \]  
(20)

Subtracting (19) from (20) we obtain
\[ (a\gamma y\beta c - c\beta y\gamma a)\beta z\gamma a = 0. \]  
(21)

which alongwith semiprimeness of \( M \) implies
\[ a\gamma y\beta c = c\beta y\gamma a \]  
(22)

Using (22), from (15) we get
\[ a\gamma y\beta (b + c) = 0. \]

In other words
\[ \left[ f(x), x^r \right]_\beta y\beta f(x), x^r \beta x^r \right]_\gamma = 0. \]  
(23)

The last relation gives
\[ \left[ f(x), x^r \right]_\beta y\gamma x^r + x^r y\beta f(x), x^r \beta x^r \right]_\gamma = 0. \]

Which implies
\[ f(x), x^r \beta x^r \right]_\gamma y\beta f(x), x^r \beta x^r \right]_\gamma = 0. \]

That is,
\[ f(x), x^r \beta x^r \right]_\gamma y\beta f(x), x^r \beta x^r \right]_\gamma = 0. \]

Semiprimeness of \( M \) implies
\[ f(x), x^r \beta x^r \right]_r = 0 \]  
(24)

**Theorem 2.2** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring with involution satisfying \( x\gamma y\alpha z = x\alpha y\gamma z \)
and \( f : M \rightarrow M \) an additive mapping such that
\[ 2[f(x), x\beta x] = f(x), x\beta x + x\gamma \beta f(x) \]
for all \( x \in M, \; \beta \in \Gamma, \) then \( f \) is a Jordan \( \gamma \)-centralizer.

**Proof.** By Lemma 2.1, linearizing (24), we get
\[ f(x), y\beta y + f(x), x\beta x + x\beta y + y\beta x + f(y), x\beta y + y\beta x \]
\[ + [f(y), x^r \beta y + y^r \beta x]_\gamma = 0. \]

Replacing \( x \) by \( -x \) in the last relation and comparing the relation so obtained with the last relation alongwith 2-torsionfreeness of \( M \), we get
\[ f(x), x^r \beta y + y^r \beta x]_\gamma + [f(y), x^r \beta x]_\gamma = 0 \]  
(25)

Replacing \( y \) by \( 2(y\beta x + x\beta y) \) in the last relation and then using (2) and (24), we get
\[ 0 = 2[f(x), (x^r \beta)^2 y^r + y^r (\beta x^r)^2 + 2x^r \beta y x^r]_\gamma \]
\[ + [f(y), x^r \beta x + y^r \beta f(x) + f(x), \beta y + x^r \beta f(y), x^r \beta x]_\gamma \]
\[ = 2(x^r \beta)^2 [f(x), y^r]_\gamma + 2[f(x), y^r]_\gamma (\beta x^r)^2 \]
\[ + 4[f(x), x^r \beta y \beta x^r]_\gamma + f(x), \beta f(y, x^r \beta x^r]_\gamma \]
\[ + x^r \beta [f(y), x^r \beta x^r]_\gamma + [f(y), x^r \beta x^r]_\gamma, \beta x^r \]
\[ + [y^r, x^r \beta x^r]_\gamma, \beta f(x). \]

That is,
\[ 2(x^r \beta)^2 [f(x), y^r]_\gamma + 2[f(x), y^r]_\gamma (\beta x^r)^2 \]
\[ + 4[f(x), x^r \beta y \beta x^r]_\gamma + f(x), \beta f(y, x^r \beta x^r]_\gamma \]
\[ + x^r \beta [f(y), x^r \beta x^r]_\gamma + [f(y), x^r \beta x^r]_\gamma, \beta x^r \]
\[ + [y^r, x^r \beta x^r]_\gamma, \beta f(x) = 0 \]  
(26)

Replacing \( y \) by \( x \) in the last relation and using 2-torsionfreeness of \( M \), we obtain
\[ (x^r \beta)^2 [f(x), x^r]_\gamma + [f(x), x^r]_\gamma (\beta x^r)^2 \]
\[ + 2[f(x), (x^r \beta)^2 x^r]_\gamma = 0, \]

which gives
\[ (x^r \beta)^2 [f(x), x^r]_\gamma + 3[f(x), x^r]_\gamma (\beta x^r)^2 = 0 \]  
(27)

From (24) we get
\[ [f(x), x^r]_\gamma, \beta x^r + x^r \beta f(x), x^r]_\gamma = 0 \]  
(28)

From the last relation by easy calculations one gets
\[ (x^r \beta)^2 [f(x), x^r]_\gamma = [f(x), x^r]_\gamma (\beta x^r)^2. \]

Using the last relation, from (27) alongwith 2-torsionfreeness of \( M \), we get
\[ [f(x), x^r]_\gamma (\beta x^r)^2 = 0 \]  
(29)

and
\[ (x^r \beta)^2 [f(x), x^r]_\gamma = 0 \]  
(30)

From (28) and then using (30) we attain
\[ x^r \beta [f(x), x^r]_\gamma, \beta x^r = 0 \]  
(31)
0 = 2(x^\beta)^2[f(x), y^\gamma] + 2[f(x), y^\gamma](\beta x^\gamma)^2
+4[f(x), x^\gamma \beta y^\gamma x^\gamma], + f(x)\beta y^\gamma x^\gamma, + f(x)\beta y^\gamma x^\gamma
+[y^\gamma, x^\gamma \beta x^\gamma], \beta f(x) - x^\gamma \beta [f(x), x^\gamma \beta y^\gamma + y^\gamma \beta x^\gamma],
-[f(x), x^\gamma \beta y^\gamma + y^\gamma \beta x^\gamma], \beta x^\gamma
= 2(x^\beta)^2[f(x), y^\gamma], + 2[f(x), y^\gamma], (\beta x^\gamma)^2
+4[f(x), x^\gamma \beta y^\gamma x^\gamma], + f(x)\beta y^\gamma x^\gamma, + f(x)\beta y^\gamma x^\gamma
+[y^\gamma, x^\gamma \beta x^\gamma], \beta f(x) - x^\gamma \beta [f(x), x^\gamma \beta y^\gamma + y^\gamma \beta x^\gamma],
-(x^\beta)^2[f(x), y^\gamma] - x^\gamma \beta [f(x), y^\gamma], y^\beta \beta x^\gamma
-x^\beta \beta [f(x), x^\gamma], - f(x), x^\gamma], \beta y^\gamma \beta x^\gamma
-x^\beta \beta [f(x), y^\gamma], \beta x^\gamma - [f(x), y^\gamma], (\beta x^\gamma)^2
-y^\beta [f(x), x^\gamma], \beta x^\gamma
That is,
(x^\beta)^2[f(x), y^\gamma], + [f(x), y^\gamma], (\beta x^\gamma)^2
+3[f(x), x^\gamma], \beta y^\gamma \beta x^\gamma + 3x^\gamma \beta y^\gamma \beta [f(x), x^\gamma],
+2x^\gamma \beta [f(x), y^\gamma], \beta x^\gamma + f(x)\beta [y^\gamma, x^\gamma \beta x^\gamma],
+[y^\gamma, x^\gamma \beta x^\gamma], \beta f(x) - x^\gamma \beta [f(x), x^\gamma \beta y^\gamma + y^\gamma \beta x^\gamma],
-y^\beta [f(x), x^\gamma], (\beta x^\gamma)^2 = 0\tag{33}
Subtracting \eqref{33} from \eqref{34}, we have
(x^\beta)^2 y^\beta \beta [f(x), x^\gamma], + 3x^\gamma \beta y^\gamma \beta [x^\gamma, [f(x), x^\gamma]],
+2x^\gamma \beta y^\gamma \beta [f(x), x^\gamma], \beta x^\gamma + [y^\gamma, x^\gamma \beta x^\gamma], \beta [x^\gamma, f(x)], = 0,
which alongwith \eqref{30} gives
2(x^\beta)^2 y^\beta \beta [f(x), x^\gamma], + 3x^\gamma \beta y^\gamma \beta \beta [f(x), x^\gamma]
-replacing -[f(x), x^\gamma], \beta x^\gamma by x^\gamma \beta [f(x), x^\gamma], in the last relation, we obtain
(x^\beta)^2 y^\beta \beta [f(x), x^\gamma], + 2x^\gamma \beta y^\gamma \beta \beta [f(x), x^\gamma], = 0\tag{35}
Using \eqref{24}, \eqref{29}, \eqref{30} and \eqref{31}, from \eqref{11} we get
(x^\beta)^2 y^\beta \beta \beta [f(x), x^\gamma], = 0\tag{36}
which alongwith \eqref{35}, gives \(x^\gamma \beta y^\gamma \beta \beta \beta [f(x), x^\gamma], = 0\).
That is,
(x^\beta)^2 y^\beta \beta \beta [f(x), x^\gamma], y^\gamma \beta x^\gamma \beta \beta [f(x), x^\gamma], = 0.
Replacing y by y alongwith semiprimeness of \(M\), we have
\(x^\beta \beta \beta [f(x), x^\gamma], = 0\)\tag{37}
Similarly, we have
\([f(x), x^\gamma], \beta x^\gamma = 0\)\tag{38}
Combining \eqref{1} and \eqref{38}, we get
\(f(x)\beta x = f(x)\beta x^\gamma\) and
\(f(x)\beta x^\gamma = x^\gamma \beta f(x)\) for all \(x \in M\) and \(\beta \in \Gamma\). That is, \(f\) is both a left and a right Jordan \^\gamma-centralizer. Hence \(f\)
is a Jordan \^\gamma-centralizer.

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