SOME PROPERTIES OF GLUED GRAPHS AT COMPLETE CLONE IN THE VIEW OF ALGEBRAIC COMBINATORICS

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ABSTRACT: *A glued graph at complete clone K_r is obtained from combining two graphs by identifying edges of K_r of each original graph. We investigate how to change some properties such as height, big height, Krull dimension, Betti numbers by gluing of two graphs at complete clone. We give a sufficient and necessary condition so that the glued graph of two Cohen-Macaulay chordal graphs at complete clone is a Cohen-Macaulay graph. Moreover, we present the conditions that the edge ideal of gluing of two graphs at complete clone has linear resolution whenever the edge ideals of original graphs have linear resolution. We show when gluing of two independence complexes, line graphs, complement graphs can be expressed as independence complex, line graph and complement of gluing of two graphs.*

Key Words: Glued graph, Height, Big height, Krull dimension, Projective dimension, Linear resolution, Betti number, Cohen-Macaulay.

INTRODUCTION

The concept edge ideal was first introduced by Villarreal in [23], that is, let G be a simple (no loops or multiple edges) graph on the vertex set $V(G)=\{x_1,\dots,x_n\}$ and the edge set $E(G)$. Associate to G is a quadratic square free monomial ideal $I(G)\subseteq R=k[x_1,\dots,x_n]$, with k a field, which is generated by x_ix_i such that $\{x_i,x_i\}\in E(G)$. An approach to studying combinatorial properties of a graph is to examine some of algebraic invariants of the edge ideal. Indeed, an aim of recent much research has been to create a dictionary between algebraic properties of $I(G)$ and properties of G.

In 2003, Uiyyasathain presented a new class of graphs in [19], glued graphs, that is, let G_1 and G_2 be any graphs, $H_1\subseteq G_1$ and $H_2\subseteq G_2$ be non-trivial connected and such that $H_1\cong H_2$ with an isomorphism f. The glued graph of G_1 and G_2 at H_1 and H_2 with respect to f, denoted by $G_1 \lhd_{H_2\cong_f H_1} \rhd G_2$, is as the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 in the glued graph. In [15], Promsakon and Uiyyasathain characterized graph gluing between trees, forests, and bipartite graphs. Also, they could give an upper bound of the chromatic number of glued graphs in terms of their original graphs. In [21], Uiyyasathain and Saduakdee studied the perfection of glued graphs at K_n-clone. In [20], Uiyyasathain and Jongthawonwuth obtained bounds of the clique partition numbers of glued graph at K_2-clones and K_3-clones in terms of their original graphs. In [14], Pimpasalee and Uiyyasathain investigated bounds of clique covering numbers of glued graphs at K_n-clones in terms of their original graphs.

As mentioned above, the study of glued graphs from combinatorial points of view has become an active area, but our main purpose of the current paper is to express algebraic features of glued graphs at complete clone using combinatorial properties. Also, we have tried as much as possible to give an accurate description of some properties of a glued graph in terms of their original graphs. Furthermore, we intend to verify whether the property of being Cohen-Macaulay, Gorenstein (for chordal graphs) having linear resolution transfer from the glued graph to original graphs and vice versa. One of the main reasons for the importance of gluing of two graphs is the fact that this operation creates a larger class of graphs which one can obtain the results on the larger graph according to the information of the smaller graphs.

Our paper is organized as follows. In section 2, we give an explicit formula for computing the height of glued graph at complete clone. Also, we present a lower bound for the big height of the glued graph at complete clone and characterize the glued graphs satisfying such bound. In section 3, we provide a necessary and sufficient condition which the equality
$\beta_{i,i+2}\left(I\left(G_1 \lhd_{K_r} \rhd G_2\right)\right)=\beta_{i,i+2}(I(G_1))+\beta_{i,i+2}(I(G_2))-\beta_{i,i+2}(I(K_r))$
holds for any $1\leq i\leq max_{v\in V(K_r)}\{deg_{G_1\lhd_{K_r}\rhd G_2}(v)\}-1.$ We obtain explicit formulas for computing $\beta_{2,4}(G_1 \lhd_{K_r} \rhd G_2)$, $\beta_{3,6}(G_1 \lhd_{K_r} \rhd G_2)$ and also we present a lower bound for $\beta_{i,2(i+1)}(I(G_1 \lhd_{K_r} \rhd G_2))$. We show that having linear resolution of the edge ideal of glued graph at complete clone implies that edge ideals of original graphs have linear resolution. A simple example illustrates having linear resolution the edge ideals of original graphs does not guarantee the existence of this property of the edge ideal of glued graph at complete clone, then we present a necessary and sufficient condition for having linear resolution of the edge ideal of glued graph at complete clone. For chordal graphs, we provide the conditions that being Cohen-Macaulay preserves under operation gluing of two graphs at complete clone and vice versa. As a result, we obtain a necessary and sufficient condition for being Gorenstein of gluing of two Cohen-Macaulay chordal graphs at complete clone. In Section 4, we give an upper bound for the projective dimension and Alexander dual of the edge ideal of the glued graph at any clone. For any two connected graphs containing a connected subgraph H, we investigate the relation between the complement, line graph and independence complex of the glued graph at clone H and the complement, line graph and independence complex of original graphs. Furthermore, we determine a sufficient condition for vertex decomposability of the glued graph at any clone. In [16], it is proved that the glued graph of connected chordal graphs is chordal. The converse is not true in general. We characterize a useful condition for being chordal of original graphs when the glued graph is chordal.

HEIGHT AND BIG HEIGHT

Let $\boldsymbol{G}$ be a simple (no loops, multiple edges) graph with the vertex set $\boldsymbol{V(G)} = \{\boldsymbol{x_1}, \dots, \boldsymbol{x_n}\}$ and the edge set $\boldsymbol{E(G)}$. The edge ideal of $\boldsymbol{G}$ is generated by $\boldsymbol{x_i x_j}$ where $\{\boldsymbol{x_i}, \boldsymbol{x_j}\} \in \boldsymbol{E(G)}$. The complete graph on $\boldsymbol{n}$ vertices, denoted by $\boldsymbol{K_n}$, is the graph with edge set $\{\{\boldsymbol{x_i}, \boldsymbol{x_j}\}: \boldsymbol{x_i}, \boldsymbol{x_j} \in \boldsymbol{V(K_n)}, \boldsymbol{x_i} \neq \boldsymbol{x_j}\}$.

Let G_1 and G_2 be any two graphs with disjoint vertex sets. Let H_1 and H_2 be non-trivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with an isomorphism f. We combine G_1 and G_2 by identifying H_1 and H_2 with respect to the isomorphism f. This resulting graph is called glued graph of G_1 and G_2 at H_1 and H_2 with respect to f. We denote this glued graph by $G_1 \lhd_H \rhd G_2$ where H is the copy of H_1 and H_2 in this glued graph. We refer to H, H_1 and H_2 as the clones of the glued graph, G_1 and G_2, respectively, and refer to G_1 and G_2 as the original graphs. Thus the combined graph is also called the glued graph of G_1 and G_2 at H-clone; see [20].

In this section, we will give an exact formula for the height of the edge ideal of the glued graph of two connected graphs at complete clone. Furtheremore, we will investigate a lower bound for the big height of the edge ideal of such graph. The vertex covering number of G, $\alpha_0(G)$, is the smallest number of vertices in any minimal vertex cover. We distinguish the vertex covering number of the glued graph at complete clone K_r depending on the number of $N_{G_i}(v)$ appeared in minimal vertex cover of $G_i \backslash K_r$ with minimum cardinality, for any $v \in K_r$ and $i = 1,2$.

Theorem 2.1 Let G_1 and G_2 be any two graphs containing subgraph K_r. Then
$$\alpha_0(G_1 \lhd_{K_r} \rhd G_2) = \alpha_0(G_1 \backslash K_r) + \alpha_0(G_2 \backslash K_r) + r - 1$$
if and only if there exists $v \in V(K_r)$ such that $N_{G_1}(v) \subseteq A_1$ and $N_{G_2}(v) \subseteq A_2$ where A_i's are minimal vertex covers of $G_i \backslash K_r$ with $|A_i| = \alpha_0(G_i \backslash K_r)$ for $i = 1,2$. Otherwise,
$$\alpha_0(G_1 \lhd_{K_r} \rhd G_2) = \alpha_0(G_1 \backslash K_r) + \alpha_0(G_2 \backslash K_r) + r.$$

Proof. $\Rightarrow$) Since there exists a vertex $v \in V(K_r)$ that does not appear in minimal vertex cover of $G_1 \lhd_{K_r} \rhd G_2$ with minimum cardinality, it follows that all neighbourhoods of v in $G_1 \backslash K_r$ and $G_2 \backslash K_r$ will be used in minimal vertex cover of $G_i \backslash K_r$ of cardinality $\alpha_0(G_i \backslash K_r)$ for $i = 1,2$.

$\Leftarrow$) Any vertex cover of K_r contains of at least $r - 1$ vertices. Under the circumstances, K_r can be covered by $r - 1$ vertices in $G_1 \lhd_{K_r} \rhd G_2$. Hence we get
$$\alpha_0\big(G_1 \lhd_{K_r} \rhd G_2\big) = \alpha_0(G_1 \backslash K_r) + \alpha_0(G_2 \backslash K_r) + r - 1,$$
as required.

Applying previous theorem and [24, Corollary 6.1.18] and [22, Proposition 7.2.5] yields the following equality.

Corollary 2.2 Let G_1 and G_2 be any two graphs containing subgraph K_r. Then
$$dim(G_1 \lhd_{K_r} \rhd G_2) = dim(G_1 \backslash K_r) + dim(G_2 \backslash K_r) + 0 \text{ or } 1.$$

Let e, e' be two distinct edges of G. The distance between e and e' in G, denoted by $dist_G(e, e')$, is defined by the minimum length l among sequences $e_0 = e, e_1, \dots, e_l = e'$ with $e_{i-1} \cap e_i \neq \phi$, where $e_i \in E_G$. If there is no such a sequence, we define $dist_G(e, e') = \infty$. We say that e and e' are 3-disjoint in G if $dist_G(e, e') \geq 3$. A subset $E \subset E_G$ is said to be pairwise 3-disjoint if every pair of distinct edges $e, e' \in E$ are 3-disjoint in G; see [7, Definitions 2.2 and 6.3]

The graph B with $V(B) = \{w, z_1, \dots, z_d\}$ and $E(B) = \{\{w, z_i\}: i = 1, \dots, d\}$ $(d \geq 1)$ is called a bouquet. Then the vertex w is called the root of B, the vertices z_i flowers of B, and the edges $\{w, z_i\}$ stems of B; see [26, Definition 1.7]. We set

$$\boldsymbol{F(B)} \coloneqq \{z \in V(G): z \text{ is a flower of some bouquet in } \boldsymbol{B}\},$$

$$\boldsymbol{R(B)} \coloneqq \{w \in V(G): w \text{ is a root of some bouquet in } \boldsymbol{B}\},$$

$$\boldsymbol{S(B)} \coloneqq \{s \in E(G): s \text{ is a stem of some bouquet in } \boldsymbol{B}\}.$$

The type of **B** is defined by $(|F(\boldsymbol{B})|, |R(\boldsymbol{B})|)$; see [12].

Let us recall the concept semi-strongly disjoint that was introduced by Kimura in [12, Definition 5.1]. A set $B = \{B_1, B_2, \dots, B_j\}$ of bouquets of G is said to be semi-strongly disjoint in G if the following conditions are satisfied:

1). $V(B_k) \cap V(B_l) = \phi$ for all $k \neq l$.

2). Any two vertices belonging to $R(B)$ are not adjacent in G.

We set
$$d'_G = \max\{|F(B)|: B \text{ is a semi-strongly disjoint set of bouquets of } G\}.$$

Definition 2.3 (22, Definition 7.7.23) Let G be a graph. The cardinality of the largest minimal vertex cover of G is called the big height of $\mathrm{I(G)}$.

In [4, Theorem 3.3], N. Erey showed the following equality:

Lemma 2.4 For any simple hypergraph $\mathcal{H}$, the equality $\mathrm{bightI}(\mathcal{H}) = \mathrm{d'}_{\mathcal{H}}$ holds.

Theorem 2.5 Let G_1 and G_2 be any two graphs containing subgraph K_r. Then
$$deg_{G_1 \lhd_{K_r} \rhd G_2} v' + bightI(G_1 \backslash N_{G_1}[v']) + bightI(G_2 \backslash N_{G_2}[v']) \leq bightI(G_1 \lhd_{K_r} \rhd G_2),$$
which v' is the vertex of K_r with maximum degree in $G_1 \lhd_{K_r} \rhd G_2$.

Proof. Using Lemma 2.4, it suffices to construct a semi-strongly disjoint set B of bouquets of $G_1 \lhd_{K_r} \rhd G_2$. Assume that B' is a bouquet with $V(B') = \{v', z_1, \dots, z_d\} = N_{G_1 \lhd_{K_r} \rhd G_2}[v']$ and $E(B') = \{\{v', z_i\}: i = 1, \dots, d\}$ where v' is the vertex of K_r with maximum degree in $G_1 \lhd_{K_r} \rhd G_2$. Setting $B' \in B$, any vertices of $N_{G_1}[v']$ and $N_{G_2}[v']$ can belong to no bouquets other than B' in B, by definition. Suppose that B_i is the semi-strongly disjoint set of bouquets of $G_i \backslash N_{G_i}[v']$ with the maximum cardinality of flowers for $i = 1,2$. Put $B = B' \cup B_1 \cup B_2$. Then
$$deg_{G_1 \lhd_{K_r} \rhd G_2} v' + bightI(G_1 \backslash N_{G_1}[v']) + bightI(G_2 \backslash N_{G_2}[v']) = |F(B)|,$$
hence one derives the required inequality.

Theorem 2.6 [16, Theorem 2.2] Let G_1 and G_2 be connected chordal graphs and H be a connected subgraph of G_1 and G_2. Then $G_1 \lhd_H \rhd G_2$ is a chordal graph.

Example 2.7 (The sharpness of the lower bound in Theorem 2.5) Put $G = G_1 \lhd_{K_r} \rhd G_2$.

Suppose that G be a pseudo- complete graph. The notation pseudo- complete graph was introduced by authors in [16]. In such case $bight(I(G)) = pd(G)$, applying [12, Corollary 5.6] which the value is precisely computed in [16, Theorem 4.8]. Furthermore, S. Jacques was presented an explicit formula for computing the projective dimension of the complete graphs in [11, Corollary 4.2.9]. Therefore, by placing the amounts in Theorem 2.5, the desired equality follows.

• Suppose that G be a lollipop graph where the (m,n)-lollipop graph $L_{m,n}$ is a graph obtained by joining the complete graph K_m to the path graph P_n by a bridge. The authors computed the projective dimension of such graphs in [17, Theorem 4.3]. Also, using [11, Corollary 7.7.35], one can obtain the projective dimension of the path graphs. By placing the values in Theorem 2.5, one derives the equality.

Theorem2.8 *(Auslander and Buchsbaum)* [13,Theorem 19.1] Let A be a Notherian local ring and $M \neq 0$ a finite A-module. Suppose that $pdM < \infty$; then

$$pdM + depthM = depthA.$$

Corollary 2.9 Let G_1 and G_2 be chordal graphs containing subgraph K_r. Then

$$deg_{G_1 \lhd_{K_r} \rhd G_2} v' + n + m - r + depth(G_1 \lhd_{K_r} \rhd G_2) \leq depth(G_1 \backslash N_{G_1}[v']) + depth(G_2 \backslash N_{G_2}[v']),$$

which v' is the vertex of K_r with maximum degree in $G_1 \lhd_{K_r} \rhd G_2$.

Proof. Applying Theorems 2.5, 2.6, 2.8 and [12, Corollary 5.6], we obtain the desired inequality.

LINEAR RESOLUTION AND COHEN-MACAULAYNESS

Let I be a monomial ideal in a polynomial ring $R = k[x_1, \dots, x_n]$. Then we can associate to I a minimal graded free resolution of the form

$$0 \to \bigoplus_j R(-j)^{\beta_{l,j}} \to \bigoplus_j R(-j)^{\beta_{l-1,j}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}} \to R \to 0$$

where $l \leq n$ and $R(-j)$ is the R-module obtained by shifting the degrees of R by j. The number $\beta_{i,j}$ is called the ijth graded Betti number of I.

If d is the smallest degree of a generator of ideal I, then the Betti numbers $\beta_{i,i+d}(I)$ form the so called linear stand of I. An ideal I generated by elements all of degree d is said to have a linear resolution if $\beta_{i,j}(I) = 0$ for all $j \neq i + d$.

The regularity of I, denoted by $reg(I)$, is defined by

$$reg(I) := max\{j - i | \beta_{i,j}(I) \neq 0\}.$$

The projective dimension of I, denoted by $pd(I)$, is defined by

$$pd(I) := max\{i | \beta_{i,j}(I) \neq 0\}.$$

Lemma 3.1 Let G_1 and G_2 be any two graphs containing subgraph K_r and $S \subseteq V(G_1 \lhd_{K_r} \rhd G_2)$ such that $S \cap V(G_1) \neq \phi$ and $S \cap V(G_2) \neq \phi$. If $(G_1 \lhd_{K_r} \rhd G_2)_S^c$ is a disconnected subgraph, then $S \cap V(K_r) \neq \phi$.

Proof. Suppose that $S \cap V(K_r) = \phi$. Thus one can write $S = S_1 \cup S_2$ where $S_1 \subseteq V(G_1 \backslash K_r)$ and $S_2 \subseteq V(G_2 \backslash K_r)$ and $S_1, S_2 \neq \phi$. Since any vertex of S_1 is adjacent to all vertices of S_2 in $(G_1 \lhd_{K_r} \rhd G_2)_S^c$, then there is a path in $(G_1 \lhd_{K_r} \rhd G_2)_S^c$ that joins v_1 and v_2, for any $v_1 \in S_1$ and $v_2 \in S_2$, which is a contradiction.

Theorem 3.2 [8, Theorem 3.2.4] Let G be a simple graph with the edge ideal $I(G)$. Then for all $i \geq 0$

$$\beta_{i,i+2}(I(G)) = \sum (|comp(G_S^c)| - 1),$$

$$\beta_{i,2(i+1)}(I(G)) = |\{H | H \text{ is an induced subgraph of } G \text{ consisting of } i+1 \text{ disjoint edges}\}|.$$

Theorem 3.3 Let G_1 and G_2 be any two graphs containing subgraph K_r. Then

$$(*) \quad \beta_{i,i+2}\big(I(G_1 \lhd_{K_r} \rhd G_2)\big) = \beta_{i,i+2}\big(I(G_1)\big) + \beta_{i,i+2}\big(I(G_2)\big) - \beta_{i,i+2}\big(I(K_r)\big)$$

for any $1 \leq i \leq max_{v \in V(K_r)}\{deg_{G_1 \lhd_{K_r} \rhd G_2}(v)\} - 1$ if and only if for any vertex $v \in V(K_r)$ one has $N_{G_1 \backslash K_r}(v) \neq \phi$ or $N_{G_2 \backslash K_r}(v) \neq \phi$, but not both.

Proof. $\Leftarrow)$ By Theorem 3.2, we may choose the set $S \subseteq V(G_1 \lhd_{K_r} \rhd G_2)$ where $|S| = i + 2$ and the number of connected components of $(G_1 \lhd_{K_r} \rhd G_2)_S^c$ is at least two. Consider the following cases:

1.Take $S \subseteq V(G_1)$.The only contribution to $\beta_{i,i+2}(I(G_1 \lhd_{K_r} \rhd G_2))$ is of $|S| = i + 2$ whenever the number of connected components of $(G_1 \lhd_{K_r} \rhd G_2)_S^c$ is at least two. Summing $|comp((G_1 \lhd_{K_r} \rhd G_2)_S^c)| - 1$ over all subsets $S \subseteq V(G_1)$ satisfying these properties, we have $\beta_{i,i+2}(I(G_1))$.

2. Take $S \subseteq V(G_2)$. Replacing G_1 by G_2 leads to similar result as that of previous case.

We claim that there is no subset $S \subseteq V(G_1 \lhd_{K_r} \rhd G_2)$ with $|S| = i + 2$ which $S \cap V(K_r) \neq \phi$, $S \cap V(G_1 \backslash K_r) \neq \phi$, $S \cap V(G_2 \backslash K_r) \neq \phi$ and $(G_1 \lhd_{K_r} \rhd G_2)_S^c$ is disconnected. Suppose that there exists a subset $S \subseteq V(G_1 \lhd_{K_r} \rhd G_2)$ that holds in situation described. Since $(G_1 \lhd_{K_r} \rhd G_2)_S^c$ is disconnected and any vertex of $S \cap V(G_1 \backslash K_r)$ is adjacent to any vertex of $S \cap V(G_2 \backslash K_r)$ in $(G_1 \lhd_{K_r} \rhd G_2)_S^c$, so we must have a vertex $v \in S \cap V(K_r)$ such that v is adjacent to all vertices of $S \cap V(G_1 \backslash K_r)$ and $S \cap V(G_1 \backslash K_r)$ in $G_1 \lhd_{K_r} \rhd G_2$, a contradiction. Also, note that by Lemma 3.1, there is no subset $S \subseteq V(G_1 \lhd_{K_r} \rhd G_2)$ which $S \cap V(K_r) = \phi$ and $(G_1 \lhd_{K_r} \rhd G_2)_S^c$ is disconnected. Therefore, summing all possibilities provides the result. Note that the subset $S \subseteq V(K_r)$ where $|S| = i + 2$ and the number of connected components of $(G_1 \lhd_{K_r} \rhd G_2)_S^c$ is at least two, is considered in both cases 1 and 2 and hence subtract $\beta_{i,i+2}(I(K_r))$ from the rest.

$\Rightarrow)$ Assume that there exist $v \in V(K_r)$ such that $|N_{G_1 \backslash K_r}(v)| \geq 1$ and $|N_{G_2 \backslash K_r}(v)| \geq 1$. Put $i = r$ and $S = V(K_r) \cup \{x, y\}$ where $x \in N_{G_1 \backslash K_r}(v)$ and $y \in N_{G_2 \backslash K_r}(v)$. Thus $(G_1 \lhd_{K_r} \rhd G_2)_S^c$ consists of two connected components, hence the equality (*) does not hold by Theorem 3.2, which is a contradiction.

Lemma 3.4 [24, page 192] Let $I = I(G) \subset R$ be the edge ideal of a graph G. If

$$\dots \to R(-4)^c \bigoplus R(-3)^b \to R(-2)^q \to R \to R/I \to 0$$

is the minimal graded resolution of R/I. The value c is equal to the number of unordered pairs of lines $\{f_i, f_j\}$ such that f_i and f_j are independent lines that cannot be joined by an edge.

Theorem 3.5 Let G_1 and G_2 be any two graphs containing subgraph K_r. *Then*

$$\beta_{2,4}(G_1 \lhd_{K_r} \rhd G_2) = \beta_{2,4}(G_1) + \beta_{2,4}(G_2) + |A_1| + |A_2| + |A_3|,$$

where

$A_1 = \{\{e_1, e_2\} | e_i \in E(G_i) \text{ and } e_i \cap V(K_r) = \phi \text{ for } i = 1,2\}$,

$A_2 = \{\{e_1, e_2\} | e_i = \{x_i, y_i\} \in E(G_i) \text{ for } i = 1,2, e_1 \cap V(K_r) = \phi, e_2 \cap V(K_r) = \{x_2\} \text{ and } \{x_2, x_1\}, \{x_2, y_1\} \notin E(G_1)\}$

$A_3 = \{\{e_1, e_2\} | e_i = \{x_i, y_i\} \in E(G_i) \text{ for } i = 1,2, e_1 \cap V(K_r) = \{x_1\}, e_2 \cap V(K_r) = \phi \text{ and } \{x_1, x_2\}, \{x_1, y_2\} \notin E(G_2)\}$

.**Proof.** To compute $\beta_{2,4}(G_1 \lhd_{K_r} \rhd G_2)$, using lemma 3.4 we need count the number of unordered pairs of edges $\{e_1, e_2\}$ such that $dist_{G_1 \lhd_{K_r} \rhd G_2}(e_1, e_2) \geq 3$. Let $e_1, e_2 \in E(G_1)$ such that $dist_{G_1}(e_1, e_2) \geq 3$. The number of the edges satisfying this property equals $\beta_{2,4}(G_1)$. Similarly, there exist $\beta_{2,4}(G_2)$ unordered pairs of edges $\{e_1, e_2\}$ of G_2 such that $dist_{G_2}(e_1, e_2) \geq 3$. Let A_1 be the set described above and $\{e_1, e_2\} \in A_1$. We claim that $dist_{G_1 \lhd_{K_r} \rhd G_2}(e_1, e_2) \geq 3$. If

$dist_{G_1 \lhd_{K_r} \rhd G_2}(e_1, e_2) = 2$, then e_1 is adjacent to e_2 by $e \in E(K_r)$. Hence, $e_1 \cap e \neq \phi$, which is a contradiction. Also, if $dist_{G_1 \lhd_{K_r} \rhd G_2}(e_1, e_2) = 1$, then $e_1 \cap e_2 \neq \phi$, this means that there exists $x \in V(K_r)$ such that $x \in e_1 \cap e_2$, a contradiction.

Now, suppose that $e_1 \in E(G_1 \backslash K_r)$ and $e_2 \in E(G_2)$ which $e_2 \cap V(K_r) \neq \phi$. Set $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$ such that $x_2 \in V(K_r)$. If there exists $\{x_1, x_2\} \in E(G_1)$ or $\{y_1, x_2\} \in E(G_1)$, then one obtains $dist_{G_1 \lhd_{K_r} \rhd G_2}(e_1, e_2) = 2$. Therefore, the contribution to $\beta_{2,4}(G_1 \lhd_{K_r} \rhd G_2)$ is of $\{e_1, e_2\}$ whenever we have $\{x_1, x_2\} \notin E(G_1)$ and $\{y_1, x_2\} \notin E(G_1)$, that the number of unordered pairs satisfying this property is equal to $|A_2|$, with the notation above. A similar statement holds if one replaces G_1
by G_2 and A_2 by A_3 in previous case. Note that if the edges $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$ which $e_1 \cap V(K_r) \neq \phi$and$e_2 \cap V(K_r) \neq \phi$,then $dist_{G_1 \lhd_{K_r} \rhd G_2}(e_1, e_2) \leq 2$. Summing all possibilities, it follows the desired formula.

Definition 3.6 (9, page 10) Two edges $\{x, y\}$ and $\{z, t\}$ of the graph G is said to be disconnected if *:*

1. $\{x, y\} \cap \{z, t\} = \phi$
2. $\{x, z\}$, $\{x, t\}$, $\{y, z\}$ and $\{y, t\}$ are not edges in G.

Theorem 3.7 Let G_1 and G_2 be any two graphs containing subgraph K_r. Then

$$\beta_{3,6}(G_1 \lhd_{K_r} \rhd G_2) = \beta_{3,6}(G_1) + \beta_{3,6}(G_2) + \beta_{2,4}(G_1 \backslash K_r).|E(G_2 \backslash K_r)| + \beta_{2,4}(G_2 \backslash K_r).|E(G_1 \backslash K_r)| + |E_1| + |A_1| + |A_2| + |B_1| + |B_2|,$$

where

$$E_1 = \{\{e_1, e_2, e_3\} | e_1 \in E(K_r), e_2 \in E(G_2), e_3 \in E(G_3) \text{ such that } \{e_1, e_2\} \text{ and } \{e_1, e_3\} \text{ are disconnected}\},$$

$$A_1 = \{\{e_1, e_2, e_3\} | e_1, e_2 \in E(G_1 \backslash K_r) \text{ and } e_3 \in E(G_2) \text{ such that } e_3 \cap V(K_r) \neq \phi, dist_{G_1}(e_1, e_2) \geq 3, \{e_1, e_3\} \text{ and } \{e_2, e_3\} \text{ are disconnected}\},$$

$$A_2 = \{\{e_1, e_2, e_3\} | e_1, e_2 \in E(G_2 \backslash K_r) \text{ and } e_3 \in E(G_1) \text{ such that } e_3 \cap V(K_r) \neq \phi, dist_{G_2}(e_1, e_2) \geq 3, \{e_1, e_3\} \text{ and } \{e_2, e_3\} \text{ are disconnected}\},$$

$$B_1 = \{\{e_1, e_2, e_3\} | e_1, e_2 \in E(G_1) \text{ and } e_3 \in E(G_2 \backslash K_r) \text{ such that } e_1 \cap V(K_r) = \phi, e_2 \cap V(K_r) \neq \phi, dist_{G_1}(e_1, e_2) \geq 3 \text{ and } \{e_2, e_3\} \text{ are disconnected}\},$$

$$B_2 = \{\{e_1, e_2, e_3\} | e_1, e_2 \in E(G_2) \text{ and } e_3 \in E(G_1 \backslash K_r) \text{ such that } e_1 \cap V(K_r) = \phi, e_2 \cap V(K_r) \neq \phi, dist_{G_2}(e_1, e_2) \geq 3 \text{ and } \{e_2, e_3\} \text{ are disconnected}\}.$$

Proof. By Theorem 3.2, we need to count the number of induced subgraphs of $G_1 \lhd_{K_r} \rhd G_2$ consisting of three disjoint edges. It is possible to take three disjoint edges only of $E(G_1)$ or only of $E(G_2)$. The number of the edges satisfying this property equals to $\beta_{3,6}(G_1)$ or $\beta_{3,6}(G_2)$.

One might consider the edges $\{e_1, e_2, e_3\}$ such that $\{e_1, e_2\} \in E(G_1 \backslash K_r)$,$e_3 \in E(G_2 \backslash K_r$ and $dist_{G_1}(e_1, e_2) \geq 3$ which the number of these edges is equal to $\beta_{2,4}(G_1 \backslash K_r)$ and also we have $e_3 \cap V(K_r) = \phi$, hence there can exist$\beta_{2,4}(G_1 \backslash K_r).|E(G_2 \backslash K_r|$ induced subgraphs consisting of three disjoint edges satisfying these properties. Replacing G_1 by G_2 leads to similar result as that of
previous case. Moreover, one can consider the edges $\{e_1, e_2, e_3\}$ such that $e_1 \in E(K_r)$, $e_2 \in E(G_1 \backslash K_r)$ and $e_3 \in E(G_2 \backslash K_r)$. Let $e_1 = \{x_1, y_1\}$, $e_2 = \{x_2, y_2\}$ and $e_3 = \{x_3, y_3\}$. If there exist the edges $\{x_1, x_2\}$, $\{x_1, y_2\}$, $\{y_1, x_2\}$, $\{y_1, y_2\}$ in G_1 or the edges $\{x_1, x_3\}$ and $\{x_1, y_3\}$, $\{y_1, x_3\}$, $\{y_1, y_3\}$ in G_2, then $\{e_1, e_2\}$ or $\{e_1, e_3\}$ are not disjoint. Therefore, in our situation, the contribution to $\beta_{3,6}(G_1 \lhd_{K_r} \rhd G_2)$ is of $\{e_1, e_2, e_3\}$ whenever the edges $\{e_1, e_2\}$ and $\{e_1, e_3\}$ are disconnected. With the notation above, the number of induced subgraphs consisting of these edges is equal to $|E_1|$.

It is possible to suppose that the edges $\{e_1, e_2, e_3\}$ such that $e_1, e_2 \in E(G_1 \backslash K_r)$ and $dist_{G_1}(e_1, e_2) \geq 3$, $e_3 \in E(G_2)$ in which $e_3 \cap V(K_r) \neq \phi$. In the situation considered, e_1, e_2 and e_3 are not necessarily disjoint. Let $e_1 = \{x_1, y_1\}$, $e_2 = \{x_2, y_2\}$ and $e_3 = \{x_3, y_3\}$ which $x_3 \in V(K_r)$. If there are not the edges $\{x_1, x_3\}$, $\{y_1, x_3\}$, $\{x_2, x_3\}$ and $\{y_2, x_3\}$ in G_1, then the induced subgraph on the vertex set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ consists of three disjoint edges. With the notation above, the number of induced subgraphs consisting of these edges is equal to $|A_1|$. Replacing G_1 by G_2 leads to similar result as that of previous case.

Now, suppose that the edges $\{e_1, e_2, e_3\}$ such that $e_1, e_2 \in E(G_1)$ and $dist_{G_1}(e_1, e_2) \geq 3$, in which $e_1 \cap V(K_r) = \phi$ and $e_2 \cap V(K_r) \neq \phi$. Let $e_1 = \{x_1, y_1\}$, $e_2 = \{x_2, y_2\}$ and $e_3 = \{x_3, y_3\}$ which $x_2 \in V(K_r)$. Using similar argument as above, we must have $\{x_2, x_3\}, \{x_2, y_3\} \notin E(G_2)$.With the notation above, the number of induced subgraphs consisting of the edges satisfying these properties is equal to $|B_1|$. Replacing G_1 by G_2 leads to similar result as that of previous case. Assuming all possibilities, our desired result follows.

Corollary 3.8 Let G_1 and G_2 be any two graphs containing subgraph K_r. Then

$$\beta_{i,2(i+1)}(I(G_1 \lhd_{K_r} \rhd G_2)) \geq \beta_{i,2(i+1)}(I(G_1)) + \beta_{i,2(i+1)}(I(G_2)) + \sum_{i=1}^{i} \beta_{i,2(i+1)} I(G_1 \backslash K_r).\beta_{i-j,2(i-j+1)} I(G_2 \backslash K_r).$$

Proof. By Theorem 3.2, we may consider induced subgraphs of $G_1 \lhd_{K_r} \rhd G_2$ containing $i+1$ disjoint edges. Since any edge of $G_1 \lhd_{K_r} \rhd G_2$ belongs to G_1 or G_2, we derive $\beta_{i,2(i+1)}(I(G_1)) + \beta_{i,2(i+1)}(I(G_2))$ by counting such induced subgraphs in G_1 or G_2. Also, it is possible to choose j disjoint edges of $G_1 \backslash K_r$ and $i+1-j$ disjoint edges of $G_2 \backslash K_r$. Applying Theorem 3.2, the number of the induced subgraphs containing j disjoint edges of $G_1 \backslash K_r$ and $i+1-j$ disjoint edges of $G_2 \backslash K_r$ is equal to $\beta_{j-1,2j}(I(G_1 \backslash K_r))$ and $\beta_{i-j,2(i-j+1)}(I(G_2 \backslash K_r))$, respectively. On the other hand, since the edges contain no vertices of K_r, then induced subgraph on the vertices contains $i+1$ disjoint edges of $G_1 \lhd_{K_r} \rhd G_2$. Hence, there are $\beta_{j-1,2j}(I(G_1 \backslash K_r)).\beta_{i-j,2(i-j+1)}(I(G_2 \backslash K_r))$ induced subgraphs of this type. Counting all the possibilities, we obtain $\sum_{j=1}^{i} \beta_{j-1,2j}(I(G_1 \backslash K_r)).\beta_{i-j,2(i-j+1)}(I(G_2 \backslash K_r))$. Note that j cannot equal to $i+1$, because it has been counted in case 1. Summing all the described cases, we obtain the desired bound.

In 1990, Fröberg presented a combinatorial characterization of the edge ideals having linear resolution.

Theorem 3.9 (Fröberg's Theorem) [2, Theorem 3.3] Let G be a graph. Then $I(G)$ has a linear resolution if and only if G^c is a chordal graph.

Theorem 3.10 Let G_1 and G_2 be any two graphs containing subgraph K_r. If $I(G_1 \lhd_{K_r} \rhd G_2)$ has a linear resolution, then $I(G_1)$ and $I(G_2)$ have a linear resolution.

Proof. Using Fröberg's Theorem, it suffices to show that G_1^c and G_2^c are induced subgraphs of $(G_1 \lhd_{K_r} \rhd G_2)^c$. Suppose that $\{x,y\} \in E(G_1^c)$, then $\{x,y\} \notin E(G_1)$. We claim that $\{x,y\} \notin E(G_1 \lhd_{K_r} \rhd G_2)$, because if not, $\{x,y\} \in E(G_1 \lhd_{K_r} \rhd G_2)$ implies that $\{x,y\} \in E(G_2)$, hence it follows that $x, y \in V(K_r)$, contradicting the fact that K_r is subgraph of G_1. In the same way, one can prove that $(G_2)^c$ is an induced subgraph of $(G_1 \lhd_{K_r} \rhd G_2)^c$.

It is natural to ask if the converse of Theorem 3.10 is true or false. That is to say, if $I(G_1)$ and $I(G_2)$ have linear resolution, then does $I(G_1 \lhd_{K_r} \rhd G_2)$ have a linear resolution?

Example 3.11 Consider $G_1 = G_2 = C_4$ and $H = K_2$. By computation by *CoCoA*, we see that $I(C_4)$ has the minimal graded free resolution as:

$$0 \to R(-4) \to R(-3)^4 \to R(-2)^4 \to R \to 0.$$

Then $I(G_1)$ and $I(G_2)$ have linear resolution. Also, by computation we deduce that $I(C_4 \lhd_{K_2} \rhd C_4)$ has the minimal graded free resolution as:

$$0 \to R(-6) \to R(-4)^4 \oplus R(-5)^2 \to R(-3)^{10} \oplus R(-4) \to R(-2)^7 \to R \to 0.$$

It follows that $I(C_4 \lhd_{K_2} \rhd C_4)$ does not have a linear resolution.

The example given above shows that the converse of Theorem 3.10 is not true in general.

Theorem 3.12 Let G_1 and G_2 be any two graphs containing subgraph K_r. Suppose that $I(G_1)$ and $I(G_2)$ have linear resolution. Then $I(G_1 \lhd_{K_r} \rhd G_2)$ has a linear resolution if and only if the following conditions are satisfied:

(i) There exist no edges $e_1 \in E(G_1 \backslash K_r)$ and $e_2 \in E(G_2 \backslash K_r)$ which are disconnected.

(ii) There exist no edges $e_1 \in E(G_1 \backslash K_r)$ and $e_2 = \{x,y\} \in E(G_2)$ which $x \in V(G_2 \backslash K_r)$ and $y \in V(K_r)$ such that e_1 and e_2 are disconnected.

(iii) There exist no edges $e_1 \in E(G_2 \backslash K_r)$ and $e_2 = \{z,t\} \in E(G_1)$ which $z \in V(G_1 \backslash K_r)$ and $t \in V(K_r)$ such that e_1 and e_2 are disconnected.

Proof. $\Rightarrow$) Assume that condition (i) does not satisfy. Thus there exist two edges $e_1 = \{x,y\} \in E(G_1 \backslash K_r)$ and $e_2 = \{z,t\} \in E(G_2 \backslash K_r)$ which are disconnected. Since $\{x,z\}$, $\{x,t\}$, $\{y,z\}$ and $\{y,t\}$ are edges in $(G_1 \lhd_{K_r} \rhd G_2)^c$, then they form a cycle of length four in $(G_1 \lhd_{K_r} \rhd G_2)^c$, a contradiction to our hypothesis that $I(G_1 \lhd_{K_r} \rhd G_2)$ has a linear resolution. The conditions (ii) and (iii) follow using similar argument.
$\Leftarrow$) We claim that there exist no cycle without chord C_n of length $n \geq 5$ in $(G_1 \lhd_{K_r} \rhd G_2)^c$. Being Chordal G_1^c and G_2^c guarantees to the existence of at least a vertex $x_1 \in V(G_1 \backslash K_r)$ and a vertex $x_2 \in V(G_2 \backslash K_r)$ in C_n which are adjacent. Further vertex, x_3, can belong to only $V(K_r)$ or $V(G_1 \backslash K_r)$, because if $x_3 \in V(G_2 \backslash K_r)$ then C_n has a chord $\{x_1,x_3\}$. Consider $x_3 \in V(K_r)$. If $x_4 \in V(G_1 \backslash K_r)$ implies that $\{x_2,x_4\}$ be a chord. Otherwise, $x_4 \in V(G_2 \backslash K_r)$, hence we have $\{x_1,x_4\} \in E(C_n)$ which means $n = 4$. Now assume that $x_3 \in V(G_1 \backslash K_r)$. If $x_4 \in V(G_1 \backslash K_r)$ then $\{x_2,x_4\}$ is a chord. Suppose that $x_4 \in V(G_2 \backslash K_r)$. It implies that $\{x_1,x_4\} \in E(C_n)$, hence $n = 4$. If $x_4 \in V(K_r)$ then it may happen $x_5 \in V(G_1 \backslash K_r)$ or $x_5 \in V(G_2 \backslash K_r)$. In the case that $x_5 \in V(G_1 \backslash K_r)$, it follows that x_2 is adjacent to x_5 and in the case $x_5 \in V(G_2 \backslash K_r)$, x_3 and x_1 are adjacent to x_5, which proves the claim.

Now, it remains to show that there exists no cycle of length four in $(G_1 \lhd_{K_r} \rhd G_2)^c$. Suppose that $(G_1 \lhd_{K_r} \rhd G_2)^c$ contains the cycle C_4. Since all of the vertices of K_r are adjacent in $G_1 \lhd_{K_r} \rhd G_2$, then no two vertices of K_r can join in $(G_1 \lhd_{K_r} \rhd G_2)^c$. On the other hand, by assumption G_1^c and G_2^c are chordal graphs, hence C_4 contains at least a vertex of $G_1 \backslash K_r$ and a vertex of $G_2 \backslash K_r$. Assume that there are two non-adjacent vertices of K_r in C_4. It follows that the verices of $G_1 \backslash K_r$ and $G_2 \backslash K_r$ are adjacent in $G_1 \lhd_{K_r} \rhd G_2$, a contradiction. Consider the case that there is unique vertex of K_r in C_4. Suppose that two non-adjacent vertices of C_4 belong to $V(G_1 \backslash K_r)(or\ V(G_2 \backslash K_r))$ which means there exist two disconnected edges. Thus we reach a contradiction to condition (ii)(condition (iii)). If there exists no vertex of K_r in C_4, it implies that we have two disconnected edges, a contradiction to condition (i). This completes the proof.

A simplicial complex Δ on the vertex set V is a collection of subsets of V such that $G \subseteq F$ and $F \in \Delta$ implies $G \in \Delta$. A free vertex is a vertex which belongs to exactly one facet. A stable set or clique of G is a subset F of $V(G)$ such that $\{i,j\} \in E(G)$ for all $i, j \in F$ with $i \neq j$. We write $\Delta(G)$ for the simplicial complex on $V(G)$ whose faces are the stable subsets of G. The graph G is called Cohen-Macaulay(Gorenstein) over Kif $K[x_1, \ldots, x_n]/I(G)$ is a Cohen-Macaulay (Gorenstein) ring, and is called Cohen-Macaulay (Gorenstein) if it is Cohen-Macaulay(Gorenstein) over any field; see [10].

Herzog, Hibi and Zheng classify all Cohen-Macaulay chordal graphs as follows.

Theorem 3.13 [10] Let K be a field, and let G be a chordal graph on the vertex set $[n]$. Let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which admit a free vertex. Then the following conditions are equivalent:

1. G is Cohen-Macaulay;
2. G is Cohen-Macaulay over K;
3. G is unmixed;
4. $[n]$ is the disjoint union of $F_1, \ldots, F_m$.

Theorem 3.14 Let G_1 be a chordal graph on the vertex set $[n]$, and let $F_1, \ldots, F_\alpha$ be the facets of $\Delta(G_1)$ which admit a free vertex. Also assume that G_2 be a chordal graph on the vertex set $[m]$ and $F'_1, \ldots, F'_\beta$ be the facets of $\Delta(G_2)$ which admit a free vertex. If G_1 and G_2 be Cohen-Macaulay then $G_1 \lhd_{K_r} \rhd G_2$ is Cohen-Macaulay if and only if there exist $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$ such that $K_r = F_i = F'_i$ or $K_r = F_i \subseteq F'_i$ or $K_r = F'_i \subseteq F_i$ and other facets of $\Delta(G_1)$ and $\Delta(G_2)$ containing free vertex be pairwise disjoint.

Proof. $\Rightarrow$) We proceed by contradiction assuming that $K_r \subset F_i$ and $K_r \subset F'_i$ for some $1 \leq i \leq \alpha$ and $1 \leq j \leq \beta$. Pick the facets of $\Delta(G_1 \lhd_{K_r} \rhd G_2)$. Since $G_1 \lhd_{K_r} \rhd G_2$ is Cohen-Macaulay, then the vertices of K_r belong to exactly one facet in clique complex $\Delta(G_1 \lhd_{K_r} \rhd G_2)$, by Theorem 3.13. Without loss of generality, assume that $K_r \subset F'_i$. Since $K_r \neq F_i$, there is $x \in F_i \backslash K_r$ which is not covered by any facet with free vertex of $\Delta(G_1 \lhd_{K_r} \rhd G_2)$, because G_1 is Cohen-Macaulay. Therefore, we have shown that $G_1 \lhd_{K_r} \rhd G_2$ is not Cohen-Macaulay, a contradiction.

To show the second part, suppose that there exist $F_i \in \Delta(G_1)$ and $F'_i \in \Delta(G_2)$ such that $F_i \cap F'_i \neq \phi$. Since $F_i \neq F'_i$, then there exist $x \in F_i \backslash F'_i$ and $y \in F'_i \backslash F_i$. We may assume, without loss of generality, that the vertices of $F_i \cap F'_i$ is contained in the facet F_i of $\Delta(G_1 \lhd_{K_r} \rhd G_2)$. Note that being Cohen-Macaulay of $G_1 \lhd_{K_r} \rhd G_2$ implies that the vertices of

$F_i \cap F'_i$ is contained in either F_i or F'_i. Also, no other facet of $\Delta(G_1)$ which admit a free vertex can contain y, because G_1 is Cohen-Macaulay. Hence y does not belong to any facet with free vertex in $\Delta(G_1 \lhd_{K_r} \rhd G_2)$, a contradiction.

$\Leftarrow$) It suffices to show that the disjoint union of the facets containing free vertex of $\Delta(G_1 \lhd_{K_r} \rhd G_2)$ is $[n+m-r]$, by using Theorems 2.6 and 3.13. Take an arbitrary vertex $x \in V(G_1 \lhd_{K_r} \rhd G_2)$.

1. Suppose that $x \in V(G_1 \backslash K_r)$. Since G_1 is Cohen-Macaualy, hence x belongs to exactly one facet having free vertex of $\Delta(G_1)$;

2. Suppose that $x \in V(G_2 \backslash K_r)$. Since G_2 is Cohen-Macaualy, hence x belongs to exactly one facet having free vertex of $\Delta(G_2)$;

3. Suppose that $x \in V(K_r)$. Since G_1 and G_2 are Cohen-Macaulay, then there exist $1 \le i \le \alpha$ and $1 \le j \le \beta$ such that $x \in F_i$ and $x \in F'_j$. Hence, using assumption we get $F_i = F'_j = K_r$ or $K_r = F_i \subseteq F'_j$ or $K_r = F'_j \subseteq F_i$. Therefore, the vertex x is contained in a facet which admit a free vertex in $\Delta(G_1 \lhd_{K_r} \rhd G_2)$, as required.

Corollary 3.15 Let G_1 be a chordal graph on the vertex set $[n]$, and let $F_1, \ldots, F_\alpha$ be the facets of $\Delta(G_1)$ which admit a free vertex. Also assume that G_2 be a chordal graph on the vertex set $[m]$ and $F'_1, \ldots, F'_\beta$ be the facets of $\Delta(G_2)$ which admit a free vertex. If $G_1 \lhd_{K_r} \rhd G_2$ be Cohen-Macaulay then G_1 and G_2 are Cohen-Macaulay if and only if there are $1 \le i \le \alpha$ and $1 \le j \le \beta$ such that $K_r = F_i = F'_i$ or $K_r = F_i \subseteq F'_i$ or $K_r = F'_i \subseteq F_i$ and other facets of $\Delta(G_1)$ and $\Delta(G_2)$ containing free vertex be pairwise disjoint.

Proof. $\Rightarrow$) With the same arguments as used in the proof of Theorem 3.14, one can show the desired conclusion.

$\Leftarrow$) First assume that there exist $1 \le i \le \alpha$ and $1 \le j \le \beta$ such that $K_r = F_i = F'_i$. We will show that any vertex of $G_1 \backslash K_r$ and $G_2 \backslash K_r$ is contained in a facet having free vertex of $\Delta(G_1) \backslash K_r$ and $\Delta(G_2) \backslash K_r$, respectively. Consider the vertices $x \in V(G_1 \backslash K_r)$ and $y \in V(G_2 \backslash K_r)$.Since $G_1 \lhd_{K_r} \rhd G_2$ is Cohen-Macaulay, then there is $1 \le s \ne i \le \alpha$ such that $x \in F_s$. We can repeat the same argument to obtain $y \in F'_t$ which $1 \le t \ne j \le \beta$ and by assumption $F_s \cap F'_t = \phi$. Hence G_1 and G_2 are Cohen-Macaulay, by Theorem 3.13.

Now suppose that $K_r = F_i \subseteq F'_i$ or $K_r = F'_i \subseteq F_i$. Applying the same argument, the desired result yields.

Lemma 3.16 [10, Corollary 2.1] Let G be a Cohen-Macaulay chordal graph, and let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which have a free vertex. Let i_j be a free vertex of F_j for $j = 1, \ldots, m$, and let G' be the induced subgraph of G on the vertex set $[n]$. Then G is Gorenstein, if and only if G is a disjoint union of edges.

Theorem 3.14 and Lemma 3.16 imply the following characterization of Gorenstein glued graph at complete clone:

Corollary 3.1 Let G_1 be a Cohen-Macaulay chordal graph on the vertex set $[n]$, and let $F_1, \ldots, F_\alpha$ be the facets of $\Delta(G_1)$ which admit a free vertex. Also assume that G_2 be a Cohen-Macaulay chordal graph on the vertex set $[m]$ and $F'_1, \ldots, F'_\beta$ be the facets of $\Delta(G_2)$ which admit a free vertex. If there are $1 \le i \le \alpha$ and $1 \le j \le \beta$ such that $K_r = F_i = F'_i$ or $K_r = F_i \subseteq F'_i$ or $K_r = F'_i \subseteq F_i$ and other facets of $\Delta(G_1)$ and $\Delta(G_2)$ containing free vertex be pairwise disjoint, then $G_1 \lhd_{K_r} \rhd G_2$ is Gorenstein if and only if G_1 and G_2 be Gorenstein and $r = 2$.

SOME FACTS ON GLUING OF TWO GRAPHS AT ARBITRARY CLONE

The aim of this section is to give some properties of gluing of two graphs, line graphs or independence complexes at arbitrary clone.

Lemma 4.1 [6, Lemma 2.3] Let G be a simple graph with edge ideal $I(G)$. Then

$I(G)^\vee =$

$(\{x_{i_1} \ldots x_{i_k} | \{x_{i_1}, \ldots, x_{i_k}\} \text{ is a vertex cover of } G\})$,

and the minimal generators of $I(G)^\vee$ correspond to minimal vertex covers.

Theorem 4.2 Let G_1 and G_2 be connected graphs containing a connected subgraph H. Then

$I(G_1 \lhd_H \rhd G_2)^\vee = I(G_1)^\vee \cap I(G_2)^\vee$.

emma 4.3 [23] For any graph G we have

$$reg(R/I(G)) = pd(I(G)^\vee).$$

Lemma 4.4 [5, Proposition 3.16] Let I_1 and I_2 be monomial ideals of R. Then we have:

1.$reg(R/(I_1 + I_2)) \le reg(R/I_1) + reg(R/I_2)$;

2.$reg(R/(I_1 \cap I_2)) \le reg(R/I_1) + reg(R/I_2) + 1$.

Corollary 4.5 Let G_1 and G_2 be connected graphs containing a connected subgraph H. Then we have*:* 1. $pd(I(G_1 \lhd_H \rhd G_2)) \le pd(I(G_1) + pd(I(G_2) + 1$;

2. $pd(I(G_1 \lhd_H \rhd G_2)^\vee) \le$
$pd(I(G_1)^e)^\vee + pd(I(G_2)^e)^\vee$.

Proof. 1. Applying Theorem 4.2 and Lemmas 4.3 and 4.4, we obtain

$pd(I(G_1 \lhd_H \rhd G_2)) = reg(R/I(G_1 \lhd_H \rhd G_2)^\vee) = reg(R/I(G_1)^\vee \cap I(G_2)^\vee)$
$\le reg(I(G_1)^\vee) + reg(I(G_2)^\vee) + 1$
$= pd(I(G_1)) + pd(I(G_2)) + 1$.

2. We have $I(G_1 \lhd_H \rhd G_2) = I(G_1)^e + I(G_2)^e$. Using Lemmas 4.3 and 4.4, one obtains

$pd(I(G_1 \lhd_H \rhd G_2)^\vee) = reg(R/I(G_1 \lhd_H \rhd G_2))$
$= reg(R/(I(G_1)^e + I(G_2)^e)$
$\le reg(R/(I(G_1)^e)) + reg(R/(I(G_2)^e))$
$= pd(I(G_1)^e)^\vee + pd(I(G_2)^e)^\vee$,

as required.

Let G be a graph on the vertices $x_1, \ldots, x_n$. The complement graph of G, G^c, is a graph with the same vertex set such that the vertices x_i and x_j are adjacent in G^c if and only if x_i and x_j are non-adjacent in G.

Theorem 4.6 Let G_1 and G_2 be connected graphs containing a connected subgraph H. Then the relation $(G_1 \lhd_H \rhd G_2)^c = G_1^c \lhd_{H^c} \rhd G_2^c$ holds if and only if G_2 be an induced subgraph of G_1 and $H = G_2$ or G_1 be an induced subgraph of G_2 and $\mathrm{H} = \mathrm{G}_1$.

Proof. $\Leftarrow$) Without loss of generality, we may assume that G_2 be an induced subgraph of G_1 and $H = G_2$, hence $G_1 \lhd_H \rhd G_2 = G_1$ and $(G_1 \lhd_H \rhd G_2)^c = G_1^c$. On the other hand, since G_2 is an induced subgraph of G_1, then G_2^c will be induced subgraph of G_1^c and $G_1^c \lhd_{H^c} \rhd G_2^c = G_1^c$. Hence, it follows the required equality.

$\Rightarrow$) First we show that $V(G_2) \subseteq V(G_1)$ or $V(G_1) \subseteq V(G_2)$, because if not, there are $x \in V(G_2) \backslash V(G_1)$ and $y \in V(G_1) \backslash V(G_2)$, then $e = \{x, y\} \in E((G_1 \lhd_H \rhd G_2)^c)$. On the other hand, $e \notin E(G_1^c)$ and $e \notin E(G_2^c)$ imply that $e \notin E(G_1^c \lhd_{H^c} \rhd G_2^c)$, a contradiction. Without loss of generality, we may assume $V(G_2) \subseteq V(G_1)$. To complete the proof we now show that $G_2 = < V(G_2) >$. Suppose that there exist $x, y \in V(G_2)$ such that $e = \{x, y\} \in E(G_1)$ but

$e \notin E(G_2)$. One can conclude $e \in E(G_1 \lhd_H \rhd G_2)$ then $e \notin E((G_1 \lhd_H \rhd G_2)^c)$. Also, we have that $e \in E(G_2^c)$ and $e \notin E(G_1^c)$, hence $e \in E(G_1^c \lhd_{H^c} \rhd G_2^c)$, which is a contradiction.

Assume that $H \neq G_2$, then there is $e \in E(G_2)$ such that $e \notin E(H)$ or there exists a vertex $x \in V(G_2)$ such that $x \notin V(H)$. Since G_2 is an induced subgraph of G_1, we must have $e \in E(G_1)$. It follows that $e \in E(G_1) \cap E(G_2) = E(H)$, a contradiction. If $x \in V(G_2) \backslash V(H)$, then we must have $x \in V(G_1)$. By definition of gluing, $V(G_1) \cap V(G_2) = V(H)$ but $x \notin V(H)$, a contradiction.

Definition 4.7 Let $G = (V(G), E(G))$ be a connected graph. The line graph $L(G)$ of G is the graph generated by $V(L(G)) = E(G)$ such that for any $e, f \in V(L(G))$, e is adjacent to f in $L(G)$ if and only if $e \cap f \neq \phi$.

Lemma 4.8 Let G_1 and G_2 be connected graphs containing a connected subgraph H.Then $L(G_1) \lhd_{L(H)} \rhd L(G_2)$ is a subgraph of $L(G_1 \lhd_H \rhd G_2)$.

Proof. Clearly$V(L(G_1) \lhd_{L(H)} \rhd L(G_2)) \subseteq V(L(G_1 \lhd_H \rhd G_2))$. Assume that $\{e_i, e_j\}$ be an edge of $L(G_1) \lhd_{L(H)} \rhd L(G_2)$. Then $\{e_i, e_j\} \in E(L(G_1))$ or $\{e_i, e_j\} \in E(L(G_2))$. Without loss of generality, suppose that $\{e_i, e_j\} \in E(L(G_1))$. Using definition, $e_i, e_j \in E(G_1)$ such that $e_i \cap e_j \neq \phi$, hence $e_i, e_j \in E(G_1 \lhd_H \rhd G_2)$ and $e_i \cap e_j \neq \phi$. It follows that $\{e_i, e_j\} \in L(G_1 \lhd_H \rhd G_2)$, as desired.

Theorem 4.9 Let G_1 and G_2 be connected graphs containing *a* connected subgraph H. Then $L(G_1) \lhd_{L(H)} \rhd L(G_2) = L(G_1 \lhd_H \rhd G_2)$ if and only if for any $e \in E(G_1) \backslash E(H)$ and $e' \in E(G_2) \backslash E(H)$, we have $e \cap e' = \phi$.

Proof. $\Rightarrow$) We proceed by contradiction. Suppose that there are $e \in E(G_1) \backslash E(H)$ and $e' \in E(G_2) \backslash E(H)$ such that $e \cap e' \neq \phi$, then $\{e, e'\} \in L(G_1 \lhd_H \rhd G_2)$. On the other hand, the assumption $e, e' \notin E(H)$ implies that $e, e' \notin V(L(H))$ and thus $\{e, e'\} \notin E(L(H))$. Although $e \in V(L(G_1))$ and $e' \in V(L(G_2))$, but $\{e, e'\} \notin L(G_1) \lhd_{L(H)} \rhd L(G_2)$ which is a contradiction.

$\Leftarrow$) Applying Lemma 4.8, it suffices to show that $L(G_1 \lhd_H \rhd G_2)$ is a subgraph of $L(G_1) \lhd_{L(H)} \rhd L(G_2)$. Clearly, $V(L(G_1 \lhd_H \rhd G_2)) \subseteq V(L(G_1) \lhd_{L(H)} \rhd L(G_2)$.Assume that $\{e, e'\} \in E(L(G_1 \lhd_H \rhd G_2)$ .One can conclude that $e, e' \in E(G_1 \lhd_H \rhd G_2)$ and $e \cap e' \neq \phi$. Consider the following cases:

1. $e, e' \in E(G_1)$, since $e \cap e' \neq \phi$, then $\{e, e'\} \in E(L(G_1))$. It follows that $\{e, e'\} \in E(L(G_1) \lhd_{L(H)} \rhd L(G_2))$.
2. $e, e' \in E(G_2)$, since $e \cap e' \neq \phi$, then $\{e, e'\} \in E(L(G_2))$. It follows that $\{e, e'\} \in E(L(G_1) \lhd_{L(H)} \rhd L(G_2))$.

Note that the case $e \in E(G_1) \backslash E(H)$ and $e' \in E(G_2) \backslash E(H)$ does not need to verify, because we have $e \cap e' = \phi$ using assumption. Therefore $L(G_1 \lhd_H \rhd G_2)$ is a subgraph of $L(G_1) \lhd_{L(H)} \rhd L(G_2)$, as asserted.

Let G be a graph on the vertex set $V(G)$. We can associate to G a simplicial complex, denoted by Δ_G, that its faces correspond to independence sets of G. The simplicial complex Δ_G is called independence complex of G.

A natural question is whether it is true that if H be a subgraph of G then $\Delta_H \subseteq \Delta_G$. In general, answer to this question is negative, as the following simple example shows.

Example 4.10 Let G be a graph with the vertex set $\{x, y, z\}$ and the edge set $\{xy, yz, xz\}$. Conisder H with $V(H) = \{x, y, z\}$ and $E(H) = \{xy, yz\}$. Then Δ_G is a simplicial complex that its facets are $\{x\}$, $\{y\}$ and $\{z\}$, but facets of Δ_H are as $\{x, z\}$ and $\{y\}$. Hence, Δ_H is not a subset of Δ_G.

Lemma 4.11 Let H be a subgraph of G. Then $\Delta_H \subseteq \Delta_G$ if and only if H be an induced subgraph of G.

Proof. $\Rightarrow$) Assume that there are $x, y \in V(H)$ such that $\{x, y\} \in E(G)$ but $\{x, y\} \notin E(H)$, hence $\{x, y\} \in \Delta_H$ but $\{x, y\} \notin \Delta_G$, a contradiction.

$\Leftarrow$) Suppose that F be a face of Δ_H. Then F is an independent set in H. Also, it remains an independent set in G, because if there exist $x, y \in F$ such that $\{x, y\} \in E(G)$, contradicting to the fact that H is an induced subgraph of G.

Definition 4.12 A facet complex over a finite set of vertices V is a set Δ of subsets of V, such that for all $F, G \in \Delta$, $F \subseteq G$ implies $F = G$.

Definition 4.13 [1, Definition 2.2] Let Δ be a facet complex. A sequence of facets $F_1, \ldots, F_n$ is called a path if for all $i = 1, \ldots, n-1$, we have $F_i \cap F_{i+1} \neq \phi$. We say that two facets F and G are connected in Δ if there exists a path $F_1, \ldots, F_n$ with $F_1 = F$ and $F_n = G$. Finally, we say that is connected if every pair of facets is connected.

Definition 4.14 Let Δ_1 and Δ_2 be two simplicial complexes and let $\Delta'_1 \subseteq \Delta_1$ and $\Delta'_2 \subseteq \Delta_2$ be connected simplicial complexes such that $\Delta'_1 \cong \Delta'_2$ with an isomorphism f. We define the glued simplicial complex of Δ_1 and Δ_2 at Δ'_1 and Δ'_2 with respect to f as the simplicial complex that results from combining Δ_1 with Δ_2 by identifying Δ'_1 and Δ'_2 with respect to the isomorphism f. If Δ is the copy of Δ'_1 and Δ'_2 in the glued simplicial complex, then we denote the glued simplicial complex by $\Delta_1 \lhd_\Delta \rhd \Delta_2$.

Theorem 4.15 Let G_1 and G_2 be connected graphs and H be an induced subgraph of G_1 and G_2. Then $\Delta_{G_1} \lhd_{\Delta_H} \rhd \Delta_{G_2} \subseteq \Delta_{G_1 \lhd_H \rhd G_2}$.

Proof. Suppose that F be a face of $\Delta_{G_1} \lhd_{\Delta_H} \rhd \Delta_{G_2}$. We need to verify the following cases:

1. Take $F \in \Delta_{G_1}$ but $F \notin \Delta_H$. If F not be an independent set in H, then there exist $x, y \in F$ such that $\{x, y\} \in E(H)$, a contradiction to hypothesis. Hence we obtain that $F \subseteq V(G_1 \backslash H)$. Since F is an independent set in $G_1 \backslash H$, then it remains independent in $G_1 \lhd_H \rhd G_2$. Thus $F \in \Delta_{G_1 \lhd_H \rhd G_2}$.
2. Take $F \in \Delta_{G_2}$ but $F \notin \Delta_H$. By the same reason as before, we have that $F \in \Delta_{G_1 \lhd_H \rhd G_2}$.
3. Take $F \in \Delta_H$. We claim that F will be independent in $G_1 \lhd_H \rhd G_2$. Because if there exist $x, y \in F$ which are adjacent in $G_1 \lhd_H \rhd G_2$, then $\{x, y\} \in E(G_1)$ or $\{x, y\} \in E(G_2)$, while x and y are non-adjacent vertices in H, which is a contradiction to our assumption.

Corollary 4.16 Let G_1 and G_2 be connected graphs and H be an induced subgraph of G_1 and G_2. Then $\Delta_{G_1} \lhd_{\Delta_H} \rhd \Delta_{G_2} = \Delta_{G_1 \lhd_H \rhd G_2}$ if and only if $G_1 = H$ or $G_2 = H$.

Proof. $\Rightarrow$) Suppose that $G_1 \neq H$ and $G_2 \neq H$ and there exist $v \in V(G_1) \backslash V(H)$ and $w \in V(G_2) \backslash V(H)$. Since $\{v, w\} \notin E(G_1 \lhd_H \rhd G_2)$, then $\{v, w\}$ is a face of $\Delta_{G_1 \lhd_H \rhd G_2}$. On the other hand, $\{v, w\}$ is not a face of Δ_{G_1}, because of $w \notin V(G_1)$. Also, $\{v, w\}$ is not a face of Δ_{G_2}, since $v \notin V(G_2)$. Furthermore, $\{v, w\}$ is not a face of Δ_H, because of $v, w \notin V(H)$. Therefore, $\{v, w\}$ is not a face of $\Delta_{G_1} \lhd_{\Delta_H} \rhd \Delta_{G_2}$, a contradiction.

Now, assume that there exist $e = \{x, y\} \in E(G_1)\backslash E(H)$ and $e' = \{x', y'\} \in E(G_2)\backslash E(H)$, then at least one of x or y, say x, does not belong to $V(H)$. Also, at least one of x' or y', say x', does not belong to $V(H)$, because H is an induced subgraph of G_1 and G_2. Hence we have $\{x, x'\}$ is a face of $\Delta_{G_1 \lhd_H \rhd G_2}$. On the other hand, $x \notin V(G_2)$ and $x' \notin V(G_1)$ imply that $\{x, x'\}$ is not a face of $\Delta_{G_1} \lhd_{\Delta_H} \rhd \Delta_{G_2}$, a contradiction.

$\Leftarrow$) Without loss of generality, we suppose that $G_1 = H$. It follows that $G_1 \lhd_H \rhd G_2 = G_2$ and then $\Delta_{G_1 \lhd_H \rhd G_2} = \Delta_{G_2}$. On the other hand, by Lemma 4.11 we obtain that $\Delta_{G_1} \subseteq \Delta_{G_2}$, and hence $\Delta_{G_1} \lhd_{\Delta_{G_1}} \rhd \Delta_{G_2} = \Delta_{G_2}$. It follows the required equality.

A simplicial complex Δ is recursively defined to be vertex decomposable if it has only a facet or else has some vertex v so that

1.Both $\Delta\backslash v$ and $link_\Delta v$ are vertex decomposable, and

2. No face of $link_\Delta v$ is a facet of $\Delta\backslash v$.

Recall that a simplicial vertex is a vertex v such that closed neighbourhood of v is clique. A graph G is called vertex decomposable if its independence complex is vertex decomposable.

Lemma 4.17 [25, Corollary 5.5] If G is a graph such that $G\backslash N[A]$ has a simplicial vertex for any independent set A, then G is a vertex decomposable.

Theorem 4.18 Let G_1 and G_2 be the graphs such that $G_i\backslash N_{G_i}[A_i]$ has a simplicial vertex for any independent set A_i and $i = 1,2$. If for any independent set $A_i \subseteq V(G_i)$ and $i = 1,2$, we have that $H \subseteq N[A_i]$, then $G_1 \lhd_H \rhd G_2$ is a vertex decomposable.

Proof. By Lemma 4.17, it suffices to show that for any independent set C, $(G_1 \lhd_H \rhd G_2)\backslash N[C]$ has a simplicial vertex. For any independent set C of $G_1 \lhd_H \rhd G_2$, we investigate the following cases:

1. The independent set C contains the vertices of G_1. By assumption, $G_1\backslash N[C]$ has a simplicial vertex, as v, which does not belong to $V(H)$, since $H \subseteq N[C]$. Hence one can consider v as the simplicial vertex of $G_1 \lhd_H \rhd G_2$.
2. The independent set C contains the vertices of G_2. Applying the same argument as before, it yields the desired result.
3. The independent set C can be written as $C = A_1 \cup A_2 \cup A_3$ where A_i's are independent sets in $G_1\backslash H$, $G_2\backslash H$ and H, respectively and $A_1, A_2 \neq \phi$. Since $A_3 \subseteq V(H)$ and $H \subseteq N[A_i]$ for $i = 1,2$, and also $A_1 \cup A_3$ and $A_2 \cup A_3$ are independent sets, it implies that $A_3 \subseteq A_i$ for $i = 1,2$. Hence $N[C] = N_{G_1}[A_1] \cup N_{G_2}[A_2]$. From assumption $H \subseteq N[A_i]$ for $i = 1,2$, it follows that $H \subseteq N[C]$ and $(G_1 \lhd_H \rhd G_2)\backslash N[C]$ have no vertex of H. Since there are the vertices $v_1 \in G_1\backslash N_{G_1}[A_1]$ and $v_2 \in G_2\backslash N_{G_2}[A_2]$ such that $N_{G_1}[v_1]$ and $N_{G_2}[v_2]$ are clique, then v_1 and v_2 can be considered the simplicial vertices of $(G_1 \lhd_H \rhd G_2)\backslash N[C]$, as required.

Notice that gluing of two chordal graphs is a chordal graph. The converse does not hold as the following example shows. Put $G_1 = C_4$ with the edge set $\{\{x,y\},\{y,z\},\{z,t\},\{t,x\}\}$, $G_2 = C_3$ with the edge set $\{\{x,y\},\{y,t\},\{t,x\}\}$. Let H be a subgraph of G_1 and G_2 with the edges $\{x,y\}$, $\{y,t\}$. Observe that $G_1 \lhd_H \rhd G_2$ is a chordal graph whereas G_1 is not a chordal graph.

Theorem 4.19 Let G_1 and G_2 be connected graphs containing a connected subgraph H and let $G_1 \lhd_H \rhd G_2$ be a chordal graph. Then G_1 and G_2 are chordal graphs if and only if H be an induced subgraph of G_1 and G_2.

Proof. $\Leftarrow$) Assume that G_1 is not a chordal graph, hence there exists a cycle C_n of length $n \geq 4$ in G_1. By hypothesis, gluing of G_1 and G_2 at clone H does not create a new edge. Then C_n is a cycle in $G_1 \lhd_H \rhd G_2$, contradicting to the fact that $G_1 \lhd_H \rhd G_2$ is a chordal graph.

$\Rightarrow$) Suppose that there are two vertices $x, z \in V(H)$ such that $\{x,z\} \notin E(H)$, but $\{x,z\} \in E(G_1)$ or $\{x,z\} \in E(G_2)$. Without loss of generality, assume that $\{x,z\} \in E(G_2)$. Put $G_1 = C_4$ with edges $\{x,y\}$, $\{y,z\}$, $\{z,t\}$, $\{t,x\}$ and H be a subgraph with edges $\{x,y\}$ and $\{y,z\}$, also G_2 be a chordal graph such that $\{x,z\} \in E(G_2)$. Hence $G_1 \lhd_H \rhd G_2$ is a chordal graph whereas G_1 is not a chordal graph, which is a contradiction.

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